The Microscopic basis for Maxwell's Equations

Maxwell's Equations in Macroscopic Scale

$$\nabla \times H = \boxed{J_f} + \epsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\mu_0 \frac{\partial H}{\partial t} - \mu_0 \frac{\partial M}{\partial t} \qquad B = \mu_0(H + M)$$

$$\nabla \cdot B = 0 \qquad D = \epsilon_0 E + P$$

$$\nabla \cdot \epsilon_0 E = \boxed{-\nabla \cdot P + \rho_f}$$

On a microscopic scale matter, manifested above by P and M is assumed to be currents and charges of atomic origin. The electric and magnetic fields rapidly vary from point-to-point in matter because of the rapid variation spatially of these charges & currents. Thus we represent the microscopic maxwell fields by e, b which vary on atomic scale with perhaps some longer periods due to external sources. We then average over distances long compared to atomic scales but short compared to these external scale lengths (on the order of 100μ or longer say) In time we consider time averages over periods larger than the atomic time constants (10^{-14} sec or less) but shorter than the period of external signals (greater than 10^{-12} sec) These microscopic equations are

$$\nabla \times \underline{b}/\mu_0 = \underline{i} + \epsilon_0 \frac{\partial \underline{e}}{\partial t} \qquad \nabla \times \underline{e} = -\frac{\partial b}{\partial t}$$

$$\nabla \cdot \epsilon_0 \underline{e} = \rho \qquad \nabla \cdot b = 0$$

If we then average both spatially and temporally in a short time sense as indicated above

Then since

$\nabla \times \langle \underline{e} \rangle = -\frac{\partial}{\partial t}\langle b \rangle$

and $\nabla \cdot \langle b \rangle = 0$

$$\nabla \times \frac{\langle b \rangle}{\mu_0} = \langle i \rangle + \epsilon_0 \frac{\partial \langle e \rangle}{\partial t}$$

$$\nabla \times \frac{B}{\mu_0} = J_f + \epsilon_0 \frac{\partial E}{\partial t}$$ crucial step

we identify $\langle \underline{b} \rangle$ with $\underline{B}$ and $\langle \underline{e} \rangle$ with $\underline{E}$ $\nabla \times \frac{B}{\mu_0} = \nabla \times \frac{H + M}{\mu_0} = \langle i \rangle + \epsilon_0 \frac{\partial \langle e \rangle}{\partial t}$

Thus

$\nabla \times \frac{\langle b \rangle}{\mu_0} = \langle i \rangle + \epsilon_0 \frac{\partial \underline{e}}{\partial t}$ or the average of the microscopic current density must be given by

① $\langle i \rangle = J_f + \nabla \times M + \frac{\partial P}{\partial t}$

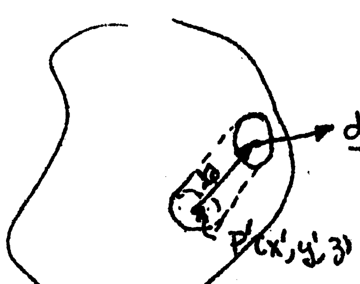
free current density; amperian current density; polarization current density

using $\nabla \cdot \langle \epsilon_0 e \rangle = \langle \rho \rangle = \nabla \cdot \epsilon_0 E = -\nabla \cdot P + \rho_f$

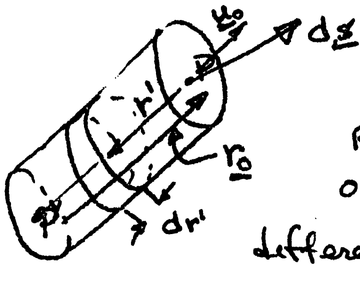
② so $\langle \rho \rangle = -\nabla \cdot P + \rho_f$

Now we must show that by suitable averaging eqns ① & ② may be found. We begin by showing ② is in fact reasonable.

We begin by calculating the charge leaving a macroscopic volume V bounded by the surface S. We choose a differential surface $d\underline{S}$ on S at the point $P(x, y, z)$. All atoms within a cylinder of length r_0 behind dS contribute a charge $\rho_a(P', r_0)\, dv_a$ to the outside of S where



$\rho_a(P', r_0)\, dv_a$ is the contribution of one atom whose charge density is $\rho_a(\underline{r}_0)$ and is located at P'. dv_a is a microscopic differential volume and ρ_a is also microscopic in nature. Now we consider only those atoms within S whose charge distribution is identical to ρ_a. (Later we will account for the various atoms of different charge density.) Looking at the cylinder more closely we see that a disc a distance r' from P contributes



$(N(P')\,dr'\,ds\,\underline{n}\cdot\underline{u}_0)\times\rho_a(P',r_0)\,dv_a$ in charge through the surface. The term ( ) in the parentheses is the total # of atoms in the disc of thickness dr, the total volume of this differential disc is ~~dr'ds~~ $dr'\,ds\,\underline{n}\cdot\underline{u}_0$ where

$\underline{u}_0 = \dfrac{\underline{r}_0}{|r_0|}$ the unit vector along $\underline{r}_0$. $N(P')$ is the number density of atoms located in the disc. It should be obvious all these atoms deposit the charge $\rho_a(P',r_0)\,dv_a$ on the outside of the surface S since r_0 is greater, in magnitude, than r'. r' extends just to r_0. any $r'>r_0$ would have atoms whose charge would not penetrate the surface S. Thus the total charge leaving this differential volume (the cylinder) is

③ → $ds\,\underline{n}\cdot\underline{u}_0\,dv_a\displaystyle\int_0^{r_0} N(P')\,\rho_a(P',r_0)\,dr' \triangleq dq(r_0)$, this is the charge leaving from all atoms with charges r_0 away from their nuclei. Later we sum up over all possible r_0.

The quantity $N(P')\,\rho_a(P',r_0)$ is slowly varying since we really are assuming many atoms within the disc. We can expand this function about the point P which is fixed by a Taylor series

④ → $N(P')\,\rho_a(P',r_0) \approx \underbrace{N(P)\,\rho_a(P,r_0)}_{\text{this is not a function of } r'} - r'\,\overbrace{\underline{u}_0\cdot\nabla_P\,(N(P')\,\rho_a(P',r_0))}^{\text{directional derivative}}$

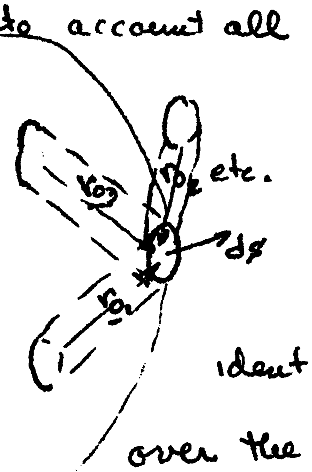
note proper sign

↑ indicates that the gradient is evaluated on the surface point P

Now using (4) in (3) we may simply integrate to find

$$dq_{s}(\underline{r_0}) = dv_a\, ds\left[\underline{n}\cdot\underline{r_0}\, \rho(P,\underline{r_0})N(P) - \tfrac{1}{2}(\underline{n}\cdot\underline{r_0})(\underline{r_0}\cdot\nabla_p(N\rho)\right]$$

Now to find the total charge leaving through ds we integrate over all $\underline{r_0}$ space. Note $dv_a = r_0^2\, dr_0 \sin\theta_a\, d\theta_a\, d\phi_a$. This really takes into account all cylinders whose base is ds and height is $|r_0|$. Then summing (integration) over all $|r_0|$ magnitudes. This is equivalent to integrating over the atom itself since all atoms have been chosen to have identical $\rho(r_0)$. After this step we integrate over the entire closed surface S giving the net charge leaving the volume within.



$$q = \oint_{\text{surface}} ds\left[\int_{\substack{\text{atom}\\ \text{volume}}} \underline{n}\cdot\underline{r_0}\, \rho(P,\underline{r_0})N(P)\, dv_a - \int_{\substack{\text{atom}\\ \text{volume}}} \tfrac{1}{2}(\underline{n}\cdot\underline{r_0})\,(\underline{r_0}\cdot\nabla_p N(P)\rho(P,\underline{r_0}))\, dv_a\right]$$

The term $\int_{\text{atom}} \underline{n}\cdot\underline{r_0}\, \rho(P,\underline{r_0})N(P)\, dv_a \triangleq \underline{P}_\theta(P)$ is the dipole moment/unit volume contributed by atoms with orientation θ (symbolic)

$P \Rightarrow x,y,z$

the second term is a higher order effect, the quadrupole moment, which is not included in Maxwell's equations — and so shows that those (Max's) equations are not complete as usually given.

Let us neglect the second term for the moment. We can see that its a small effect. Note that $\nabla_p \sim \frac{1}{L}$ where L is the macroscopic scale thus the second term is reduced by a/L from the first (a is an atomic dimension)

$$\oint_{\text{Surface}} \underline{P}_0 \cdot \underline{n}\, ds = \int_{\text{volume}} (\nabla \cdot \underline{P}_0)\, dv = q_0$$

lost to volume

thus since the volume is macroscopically small we may divide both sides by it and identify $-q_0/V_{ol}$ as $\langle \rho \rangle_0$

So $\nabla \cdot \underline{P}_0 = -\langle \rho \rangle_0$

Next we consider the sum of all atomic orientations.

If $W(\theta)\, d\theta$ is the probability of finding atomic orientation between θ & $\theta + d\theta$ then

$$\langle \rho \rangle = \int_\theta \langle \rho \rangle_0\, W(\theta) d\theta = -\nabla \cdot \underline{P} \qquad \left(\underline{P} = \int \underline{P}_0\, W(\theta) d\theta \right)$$

Now let's discuss the quadrupole term – but after a digression on vector notation.

$\underline{r}_0 = r_x \underline{a}_x + r_y \underline{a}_y + r_z \underline{a}_z$ and it is not a function of P, the macroscopic point.

$= |r_0|\, \underline{u}_0$

let $f = N\rho$ which is a function of P and r_0.

$$\underline{r}_0 (\underline{r}_0 \cdot \nabla) f = \underline{r}_0 \left[r_x \frac{\partial f}{\partial x} + r_y \frac{\partial f}{\partial y} + r_z \frac{\partial f}{\partial z} \right]$$

Now this all may be written more compactly by using tensors.

the so-called exterior product of two vectors $\underline{a}$, and $\underline{b}$ is a tensor $\underline{\underline{T}}$ defined as

$$\underline{\underline{T}} \triangleq \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix} \quad \text{thus} \quad (\underline{a}\,\underline{b})_{ik} = a_i b_k \triangleq T_{ik}$$

The operation $\nabla \cdot \underline{\underline{T}}$ is equivalent to premultiplication of the T tensor by a row matrix producing a column matrix which is a <u>vector</u>. A dot product of a vector (or vector operator like ∇) with a tensor is a vector.

$$\nabla \cdot \underline{\underline{T}} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} \triangleq \begin{bmatrix} \underline{a}_x \, \xi_x \\ \underline{a}_y \, \xi_y \\ \underline{a}_z \, \xi_z \end{bmatrix}$$

where $\xi_k = \sum_i \frac{\partial T_{ik}}{\partial x_i} \triangleq \frac{\partial T_{ik}}{\partial x_i}$ ← this is the summation notation where a sum over the repeated index $\underline{i}$ is implied.

Thus we may write $q_{ik} = \frac{1}{2}(\underbrace{\underline{r}_0 \, \underline{r}_0}^{\text{exterior product}} \rho N)_{ik}$

and $Q_{ik} = \int_{\text{atom}} q_{ik} \, dv_a$ (one of the components of the tensor $\underline{\underline{Q}}$)

and furthermore

$$\nabla_p \cdot \underline{\underline{Q}} = \frac{1}{2} \int_{\text{vol}} \underline{r}_0 \, (\underline{r}_0 \cdot \nabla_p (\rho N)) \, dv_a$$

With the above one can show

$$\langle \rho \rangle = -\nabla_p \cdot (\underline{P} - \nabla \cdot \underline{\underline{Q}})$$

quadrupole moment / volume which is a tensor.

<u>Note</u>: experimentally $\underline{\underline{Q}}$ has been measured!

Microscopic Basis of Maxwell's Equations

We wish to show

$$\langle i\rangle = \frac{1}{\Delta V} \underset{\substack{\lim \Delta V\to 0 \\ \text{in macro} \\ \text{sense}}}{\int} i\, dv = \underline{J_f} + \nabla \times \underline{M} + \frac{\partial \underline{P}}{\partial t}$$

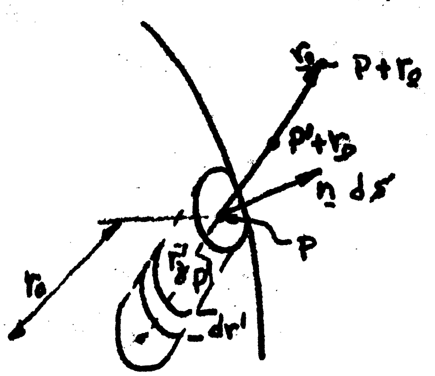
we ignore J_f since it presents no problem in that it is related to the motion of free charges $\underline{J_f} = \rho_f \underline{v}$ $\quad i_f = \rho_f v \quad \langle i_f\rangle = \langle \rho_f v\rangle = J_f$ whats left is

$$\langle i\rangle = \frac{1}{\Delta V}\int i\, dv = \frac{1}{\Delta V}\left[\sum_{\substack{\text{atoms} \\ \text{in volume}}} \int \underline{i_a}\, dv_a - \underbrace{\sum_{\ell}}_{\substack{\text{sum of atoms} \\ \text{having part} \\ \text{of its current} \\ \text{piercing sur-} \\ \text{face of } \Delta V}} \underbrace{\int i_a\, dv_a}_{\substack{\text{integral taken over} \\ \text{the parts of the} \\ \ell^{th} \text{ atom } \underline{\text{outside}} \\ \text{of volume } \Delta V}}\right]$$

call the first term $\langle i_p\rangle$, the second $\langle i_{mq}\rangle$

$$\langle i_p\rangle = \frac{1}{\Delta V} \sum_{\substack{\text{atoms} \\ \text{in volume}}} \int_{\text{atom}} i_a\, dv_a = N(P) \int_{\text{atom}} i_a\, dv_a$$

<u>$\langle i_{mq}\rangle$ is found as follows:</u>



Consider disc at P' of height dr' measured along r_o. There are $N(P') \times \underbrace{dS\, dr'\, \underline{n}\cdot\underline{u_o}}_{\text{volume of disc}}$ atoms in it. The current density contributed by all (identical) atoms in disc to $\underline{P'} + \underline{r_o}$ which is outside of volume if $r' < r_o$ is $i_a(P', \underline{r_o})$. The current due to all atoms in the disc is therefore given ~~by~~ $\underline{n}\cdot\underline{u_o}\, N(P')\, dS\, \underline{i_a}(P', \underline{r_o})dr'$.

The total current from all atoms in the differential cylinder contributing $i_a(\underline{r}_o)$ current density outside the surface is

$$di(r_o) = dS \int_0^{r_o} \underline{n}\cdot\underline{u}_o\, N(P')\, \underline{i}_a(P',r_o)dr' \quad [\text{Note: } P' = P'(r')]$$

(this is the differential current through dS)

but we may expand $N(P')\, i_a(P',r_o)$ (since it is a slowly varying function) about P $\quad N(P')\, i_a(P',\underline{r}_o) \approx N(P)\, \underline{i}_a(P,r_o) - \underbrace{(\underline{r}' \cdot \nabla)[N(P)\, \underline{i}_a(P',r_o]}_{\text{negligible}}$

thus $\quad d(i(r_o) \approx \underline{n} \cdot \underline{r}_o\, dS\, N(P)\, \underline{i}_a(P,r_o)$.

Now if we integrate over all $\underline{r}_o$ we are essentially integrating over the atomic volume. Finally all current elements through S is (divided by ΔV)

$$\langle i_{mq}\rangle = -\lim_{\Delta V\to 0}\frac{1}{\Delta V} \oint_S dS \int_{atom} (\underline{n}\cdot\underline{r}_o)\, N(P)\, \underline{i}_a(P,r_o)dv_a = -\nabla \cdot [N(P) \int_{atom} (\underline{n}\cdot\underline{r}_o)\underline{i}_a(P,r_o)dv_a]$$

Now let us examine $\langle i_p\rangle$ and $\langle i_{mq}\rangle$ to determine their physical significance

$$\langle i_p\rangle = N(P) \int_{atom} i_a\, dv_a =$$

$$\nabla \cdot i_a = -\frac{\partial\rho_a}{\partial t} \qquad \text{fast (atomic) frequencies averaged out}$$

$$\int_{\substack{all\\ vol.}} \underline{r}(\nabla \cdot \underline{i}_a)d^3r = -\int_{\substack{all\\ vol.}} \underline{i}_a\, d^3r \qquad \text{(integration by parts)}$$

We may then write

$$\langle i_p\rangle = -N(P) \int_{atom} \underline{r}_o(\nabla\cdot i)dv_a = -N(P) \frac{\partial}{\partial t} \int_{atom} \underline{r}_o\, \rho_a\, dv_a = -\frac{\partial}{\partial t} N(P)\underline{p} = -\frac{\partial}{\partial t}\underline{P}$$

In order to determine the significance of

$$<\dot{i}_{mq}> = -\frac{1}{\Delta V}\oint_S dS \int_{atom} \underline{n}\cdot\underline{r}_o \; N(P) \; \underline{i}_a(P,r_o) dv_a$$

we first recall that $\underline{\underline{q}} = \frac{1}{2} \int_{atom} \underline{r}_o \, \underline{r}_o \, \rho_a(r_o) \; dv_a$ then consider

$$I \triangleq \int_{atom} \underline{n} \cdot \underline{r}_o \, \underline{r}_o \frac{\partial\rho_a}{\partial t} dv_a = 2 \, \underline{n} \cdot \frac{\partial \underline{\underline{q}}}{\partial t}$$

but using $\nabla \cdot \underline{i}_a = -\frac{\partial\rho_a}{\partial t}$

$$I = -\int_{atom} \underline{n} \cdot \underline{r}_o \, \underline{r}_o \, \underline{\nabla} \cdot \underline{i}_a \; dv_a$$

but

$$(\underline{n} \cdot \underline{r}_o) \; \underline{\nabla} \cdot \underline{i} = \nabla \cdot \{(\underline{n} \cdot \underline{r}_o)\underline{i}\} - \underline{i} \cdot \nabla(\underline{n} \cdot \underline{r}_o)$$

direction cosines

$$\left(\begin{array}{l} n = \alpha a_x + \beta a_y + \gamma a_z \\ r_o = x' \; \underline{a}_x + y' \; \underline{a}_y + z' \; \underline{a}_z \end{array}\right)$$

radius vector

thus

$$\nabla(\underline{n} \cdot r_o) = \alpha\underline{a}_x + \beta\underline{a}_y + \gamma\underline{a}_z = \underline{n}$$

and

$$\underline{i} \cdot \nabla(\underline{n} \cdot \underline{r}_o) = \underline{i} \cdot \underline{n}$$

so

$$I = -\int_{atom} \underline{r}_o[\nabla \cdot \{(\underline{n} \cdot \underline{r}_o) \; \underline{i}\} - \underline{i} \cdot \underline{n}] \; dv_a$$

we next integrate by parts to find

$$+\frac{I}{2} = +\frac{1}{2}\int (\underline{n}\cdot\underline{r}_o)\,\underline{r}_o(\nabla\cdot i)\,dv_a = \underline{n}\cdot\frac{\partial \underline{\underline{q}}}{\partial t}$$ (follows from last eqn on p3 after integration by parts)

$$= \frac{1}{2}\int_{atom} [(\underline{n}\cdot\underline{r}_o)\underline{i} + (\underline{n}\cdot\underline{i})\underline{r}_o]\,dv_a$$ by the above analysis

$$\frac{I_-}{2} \triangleq \frac{1}{2}\int_{atom} [(\underline{n}\cdot\underline{r}_o)\underline{i} - (\underline{n}\cdot\underline{i})r_o]dv_a = \frac{1}{2}\int (\underline{r}_o\times\underline{i})\times\underline{n}\,dv_a \triangleq \underline{\mu}\times\underline{n}$$

where $\underline{\mu} \triangleq \frac{1}{2}\int_{atom} (\underline{r}_o\times\underline{i})dv_a$ and μ = atomic magnetic dipole moment where the vector identify $(a\times b)\times c = (a\cdot c)b - (\underline{b}\cdot\underline{c})\underline{a}$ has been used.

$$\frac{I}{2} + \frac{I_-}{2} = \int_{atom} (n\cdot r_o)\underline{i}\,dv_a = \underline{n}\cdot\frac{\partial \underline{\underline{q}}}{\partial t} + \underline{\mu}\times\underline{n}$$

$\underline{M}$ = [the magnetic moment/volume] = N $\underline{\mu}$

$\underline{\underline{Q}}$ = electric quadrupole moment/volume = N $\underline{\underline{q}}$

thus

$$\langle i\rangle_{mq} = \frac{1}{\Delta V}\oint_{\substack{surface\\ of\ \Delta V}} dS(\underline{n}\times\underline{M} - \underline{n}\cdot\frac{\partial \underline{\underline{Q}}}{\partial t})$$

$$\oint_S(\underline{n}\times\underline{M})dS = \int_{\Delta V} \underline{\nabla}\times\underline{M}\,dv = \lim_{\Delta V\to 0} (\Delta V)\,\nabla\times M$$ by Stoke's Theorem

and

$$-\oint dS\, n\cdot\frac{\partial \underline{\underline{Q}}}{\partial t} = -(\Delta V)\frac{\partial}{\partial t}(\nabla\cdot\underline{\underline{Q}})$$ by Gauss Theorem

finally

$$\langle i\rangle_{mq} = J_f + \frac{\partial}{\partial t}(\underline{P} - \underbrace{\nabla\cdot\underline{\underline{Q}}}_{\text{usually neglected}}) + \nabla\times\underline{M}$$

$$\nabla \times \frac{\langle b \rangle}{\mu_0} = \langle i \rangle + \epsilon_0 \frac{\partial \langle e \rangle}{\partial t}$$

correspondence.

$$\nabla \times \frac{B}{\mu_0} = J_f + \epsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t}$$

but $B = \mu_0(H + M)$

$$\therefore \quad \nabla \times (H \cdot \quad) = J_f + \epsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t}$$

$$\cancel{\nabla \times H = J_f + \epsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} - \nabla \times M}$$

but $B = \mu_0 (H+M)$

$$\frac{B}{\mu_0} = H + M$$

$$\frac{B}{\mu_0} - M = H$$

$$\therefore \quad \nabla \times \left(\frac{B}{\mu_0} - M\right) = J_f + \epsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t}$$

$$\nabla \times \frac{B}{\mu_0} = J_f + \epsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} + \nabla \times M$$

and $\langle i \rangle = J_f + \frac{\partial P}{\partial t} + \nabla \times M$

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macro: $\nabla \times E = -\frac{\partial B}{\partial t} = -\mu_0 \frac{\partial H}{\partial t} - \mu_0 \frac{\partial M}{\partial t}$

micro: $\nabla \times \langle e \rangle = -\frac{\partial \langle b \rangle}{\partial t}$

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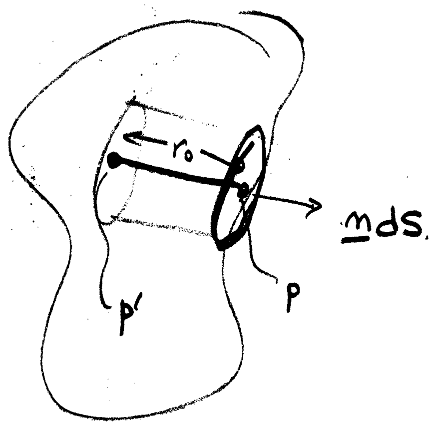
macro: $\nabla \cdot D = \rho \qquad \nabla \cdot (\epsilon_0 E + P) = \rho \qquad \nabla \cdot \epsilon_0 E = \rho_f - \nabla \cdot P$

micro: $\nabla \cdot \epsilon_0 \langle e \rangle = \varrho$

$$\varrho = \rho_f - \nabla \cdot P$$



charge is $\rho_a(p', r_0)\, dv_a$

where dv_a is a volume element at r_0 away from
and ρ_a is the charge density ~~due to~~ the atom located
at p'