EEAP 531

Horn. Chapter 3 Binary images: geometric properties

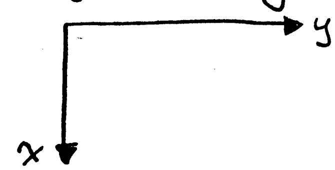
characteristic function — has a value (0 or 1) for each point in the image

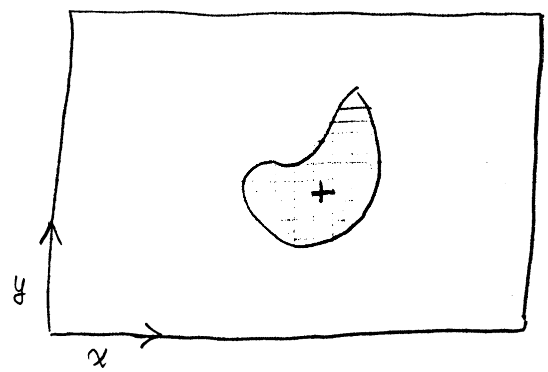
binary image —

thresholding — characteristic function = 0 if brightness > threshold (or viceversa)

set operations
point by point boolean operations } image morphology.

typical image processing format





this is not standard format

area $= \iint_I b(x,y)\,dx\,dy$ gives total area (actually zeroth moment)

position
center of area (actually center of mass) is first moment about x-axis

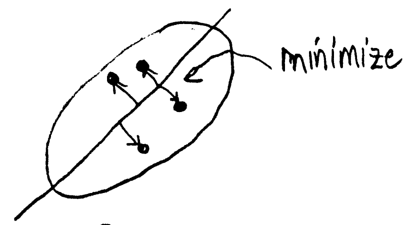
$$\bar{x} \iint_I b(x,y)\,dx\,dy = \iint x\, b(x,y)\,dx\,dy \qquad \bar{x} = \frac{\iint x\, b(x,y)\,dx\,dy}{\iint_I b(x,y)\,dx\,dy}$$

$$\bar{y} \iint_I b(x,y)\,dx\,dy = \iint y\, b(x,y)\,dx\,dy. \qquad \bar{y} = \frac{\iint y\, b(x,y)\,dx\,dy}{\iint_I b(x,y)\,dx\,dy}.$$

Some shape properties are difficult to analyze and compute

For an elongated object we will define the object orientation to be along the axis of least second moment, i.e. the axis for which the integral of the square of the distance from points in the object to the axis is a minimum, i.e. minimize

$$\text{second moment } E = \iint_I r^2 b(x,y)\,dx\,dy$$



We could do calculation as shown above but that formula is very inaccurate for object orientations parallel to the x or y axis.

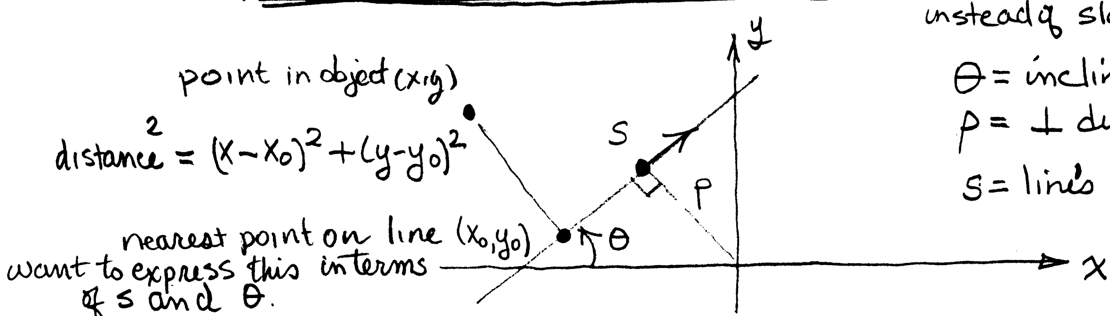
So, switch to polar coordinates

instead of slope and y-intercept

θ = inclination relative to + x-a

ρ = $\perp$ distance from origin

s = line's coordinate system



line in polar coordinates: $x\sin\theta - y\cos\theta + \rho = 0$

line in parametric form: $x_0 = -\rho\sin\theta + s\cos\theta$

$y_0 = +\rho\cos\theta + s\sin\theta$

where s is the position on the line

For each point in the object, it's distance to the line [perpendicular distance to nearest point (x_0, y_0)] is

$$r^2 = (x-x_0)^2 + (y-y_0)^2 \quad \text{substitute parametric values.}$$

$$= (x + \rho\sin\theta - s\cos\theta)^2 + (y - \rho\cos\theta - s\sin\theta)^2$$

$$= [x + (+\rho\sin\theta - s\cos\theta)]^2 + [y - (\rho\cos\theta + s\sin\theta)]^2$$

$$= x^2 + 2x\rho\sin\theta - 2xs\cos\theta + \rho^2\sin^2\theta - \cancel{2\rho s\sin\theta\cos\theta}$$
$$+ s^2\cos^2\theta + y^2 - 2y\rho\cos\theta - 2ys\sin\theta + \rho^2\cos^2\theta$$
$$+ \cancel{2\rho s\sin\theta\cos\theta} + s^2\sin^2\theta$$

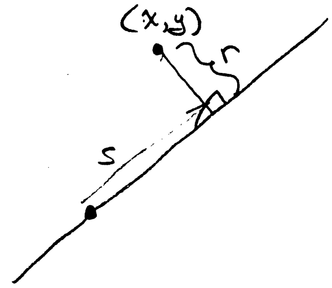
collecting terms:

$$r^2 = x^2 + y^2 + p^2 + s^2 + 2p(+x\sin\theta - y\cos\theta) + 2s(-x\cos\theta - y\sin$$

$$r^2 = x^2 + y^2 + p^2 + s^2 + 2p(x\sin\theta - y\cos\theta) - 2s(x\cos\theta + y\sin\theta)$$

This is the distance from (x,y) to a point (x_0, y_0) on the orientation axis. Now minimize the distance, i.e. what s in the line coordinate system minimizes r^2. Compute $\frac{dr}{ds}$ and set to 0.

$$2r\frac{dr}{ds} = +2s - 2(x\cos\theta + y\sin\theta) = 0$$



$$\therefore \quad x\cos\theta + y\sin\theta = s$$

Now evaluate r by substituting this result into line equations

$$x_0 = -p\sin\theta + (x\cos\theta + y\sin\theta)\cos\theta$$

$$= -p\sin\theta + x\cos^2\theta + y\sin\theta\cos\theta \qquad \text{use } \cos^2\theta = 1 - \sin^2$$

$$x_0 = -p\sin\theta + x - x\sin^2\theta + y\sin\theta\cos\theta$$

Then $x - x_0 = p\sin\theta + x\sin^2\theta - y\sin\theta\cos\theta$

$$= \sin\theta\,(p + x\sin\theta - y\cos\theta)$$

Similarily,

$$y - y_0 = \cos\theta\,(p + x\sin\theta - y\cos\theta)$$

Then, $r^2 = (x-x_0)^2 + (y-y_0)^2 = \sin^2\theta\,(p + x\sin\theta - y\cos\theta)^2$
$+ \cos^2\theta\,(p + x\sin\theta - y\cos\theta)^2$

$$r^2 = (p + x\sin\theta - y\cos\theta)^2$$

This result gives the perpendicular distance between a point (x,y) and a line (orientation) given by p, θ. Now, we minimize the total orientation function for p and θ.

Minimize $$E = \iint_I r^2 b(x,y)\,dx\,dy$$

second moment

subst. for eqn of line.

$$= \iint_I (\rho + x\sin\theta - y\cos\theta)^2 \underbrace{b(x,y)}_{\text{Restrict to binary images}}\,dx\,dy$$

$$\frac{\partial E}{\partial \rho} = \iint \cancel{2}(\rho + x\sin\theta - y\cos\theta)(1)\,b(x,y)\,dx\,dy = 0$$

drop the two.

$$= \underbrace{\rho\iint b(x,y)\,dx\,dy}_{\rho A} + \underbrace{\iint x\sin\theta\, b(x,y)\,dx\,dy}_{\sin\theta \iint x\, b(x,y)\,dx\,dy} - \underbrace{\iint y\cos\theta\, b(x,y)\,dx}_{\cos\theta \iint y\, b(x,y)\,dx\,dy}$$

$$\sin\theta\,\bar{x}\iint b(x,y)\,dx\,dy \qquad \cos\theta\,\bar{y}\iint b(x,y)\,dx\,dy$$

$$\sin\theta\,\bar{x}\,A \qquad \cos\theta\,\bar{y}\,A$$

where A = area of image

$(\bar{x},\bar{y})$ = center of mass coordinates (i.e. first moments)

$x\sin\theta - y\cos\theta + \rho = 0$

$\rho\cancel{A} + \sin\theta\,\bar{x}\cancel{A} - \cos\theta\,\bar{y}\cancel{A} = 0$

This is the equation of a line in polar coordinates.

$\Rightarrow$ First result, orientation axis passes through $(\bar{x},\bar{y})$

since this equation shows that $(\bar{x},\bar{y})$ satisfies the equation of the desired line.

<u>We will now transform coordinates to $(\bar{x},\bar{y})$ centered coordinate system and compute $\frac{\partial E}{\partial \theta}$ about center of mass</u>

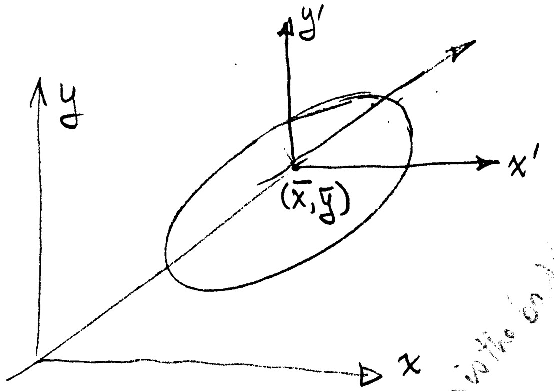
i.e. $$x' = x - \bar{x}$$
$$y' = y - \bar{y}$$

$$r = p + x\sin\theta - y\cos\theta$$
$$= p + (x' + \bar{x})\sin\theta - (y' + \bar{y})\cos\theta$$
$$= (x'\sin\theta - y'\cos\theta) + \underbrace{(p + \bar{x}\sin\theta - \bar{y}\cos\theta)}_{\text{original equation of line}} \xrightarrow{} 0 \;=\; x'\sin\theta - y'\cos\theta$$

second moment

Then, $$E = \iint (x'\sin\theta - y'\cos\theta)^2 \, b(x', y') \, dx'\, dy'$$
$$= \iint (x'^2 \sin^2\theta - 2x'y'\sin\theta\cos\theta + y'^2\cos^2\theta) \, b(x', y') \, dx'\, dy'$$
$$= \sin^2\theta \underbrace{\iint x'^2 \, b(x', y) \, dx\, dy}_{a} - 2\sin\theta\cos\theta \underbrace{\iint x'y' \, b(x', y') \, dx'\, dy'}_{b}$$
$$+ \cos^2\theta \underbrace{\iint y'^2 \, b(x', y') \, dx'\, dy'}_{c}$$

Note that a, b and c are the second moments of the object about the x', y' axes.



this will be on exam!

$$E = a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta$$

convert to 2θ

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$E = a\left(\frac{\cos 2\theta - 1}{-2}\right) - b\left(\frac{\sin 2\theta}{2}\right) + c\left(\frac{\cos 2\theta + 1}{2}\right)$$
$$= -\frac{1}{2}a\cos 2\theta + \frac{1}{2}a - \frac{1}{2}b\sin 2\theta + \frac{1}{2}c\cos 2\theta + \frac{1}{2}c$$

$$E = \frac{1}{2}(a+c) - \frac{1}{2}(a-c)\cos 2\theta - \frac{1}{2} b \sin 2\theta .$$

Now minimize with respect to θ:

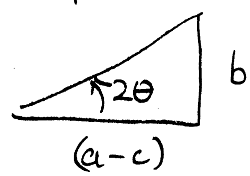
$$\frac{\partial E}{\partial \theta} = -\frac{1}{2}(a-c)(-\sin 2\theta)\,2 - \frac{1}{2} b (\cos 2\theta)\,2 = 0 .$$

$$(a-c)\sin 2\theta - b\cos 2\theta = 0$$

$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{b}{a-c}$$

$$\tan 2\theta = \frac{b}{a-c}$$

Express result geometric



by inspection

$$\sin 2\theta = \pm \frac{b}{\sqrt{b^2 + (a-c)^2}}$$

$$\cos 2\theta = \pm \frac{a-c}{\sqrt{b^2 + (a-c)^2}}$$

$\Big\}$ correct answer uses $+\sqrt{\ }$ to minimize E.

This result can be obtained in a more formal mathematical manner by seeking the rotation which diagonalizes the 2x2 matrix of second moments.

$$\begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}$$

a = x-axis moment
c = y-axis moment
b = xy-axis moment

This is <u>NOT</u> as simple as it sounds.

angular momentum matrix $\underline{H}$

see Craig
for example.

$$\underline{H} = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

The, principal axes are defined to be those for which $\underline{H}$ is diagonalized and the diagonal components of $\underline{H}$ are the principal moments of inertia, i.e.

$$\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

For two-dimensional images we have

$$\begin{bmatrix} I_{xx} & 0 \\ 0 & I_{yy} \end{bmatrix}.$$

To put into Horn's context, we want to minimze

$$\iint r^2 b(x,y)\,dx\,dy$$

The total energy of a given function is (See any text on PDE's).

$$E(u) = \int_R \left[(\nabla u)^2 + \underbrace{g(P)}_{\text{corresponds to } b(x,y)} u^2 \right] dV$$

must be a bounded positive function

For a non-trivial solution we require that u is non-zero which can be represented by $H(u) = \int_R \rho u^2\, dV = 1$ where ρ is some constant.

The solution u, which minimizes $E(u)$ is then an eigenfunction of the system with the minimum energy being given by the corresponding eigenvalue.

rewrite
$$\tan 2\theta = \frac{b}{a-c}$$

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{b}{a-c}$$

$$\therefore \quad b - b\tan^2\theta = 2(a-c)\tan\theta$$

$$\tan^2\theta + 2\frac{(a-c)}{b}\tan\theta - 1 = 0.$$

$\bar{m}_{20} = a$

$\bar{m}_{02} = c$

$\bar{m}_{11} = b.$

Hint: this equation is equivalent to

$$\left(\frac{b}{2}\tan\theta + a\right)^2 - (a+c)\left(\frac{b}{2}\tan\theta + a\right) + \left(ac - \frac{b^2}{4}\right) = 0$$

which implies that

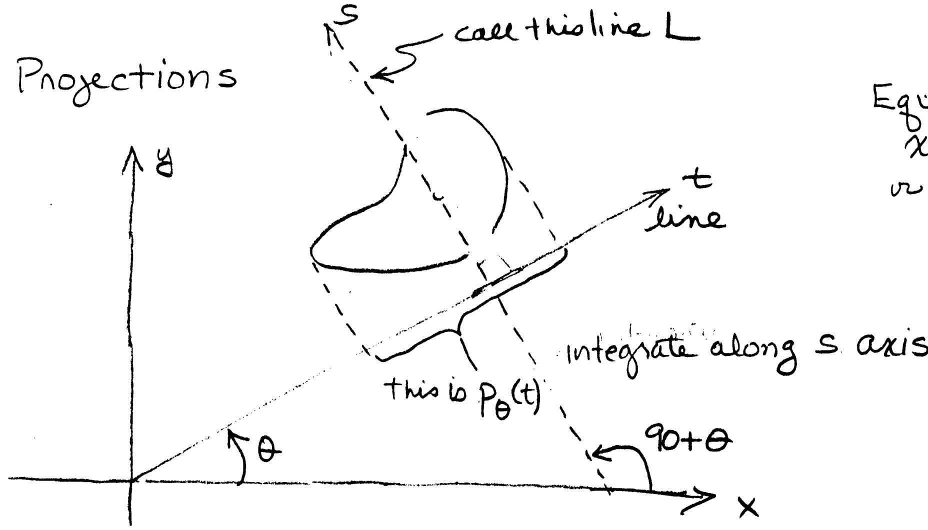
$$\frac{b}{2}\tan\theta + a$$

is an eigenvalue of the matrix

$$\begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}$$

The principal axis is then along the eigenvector corresponding to the larger eigenvalue of this matrix.

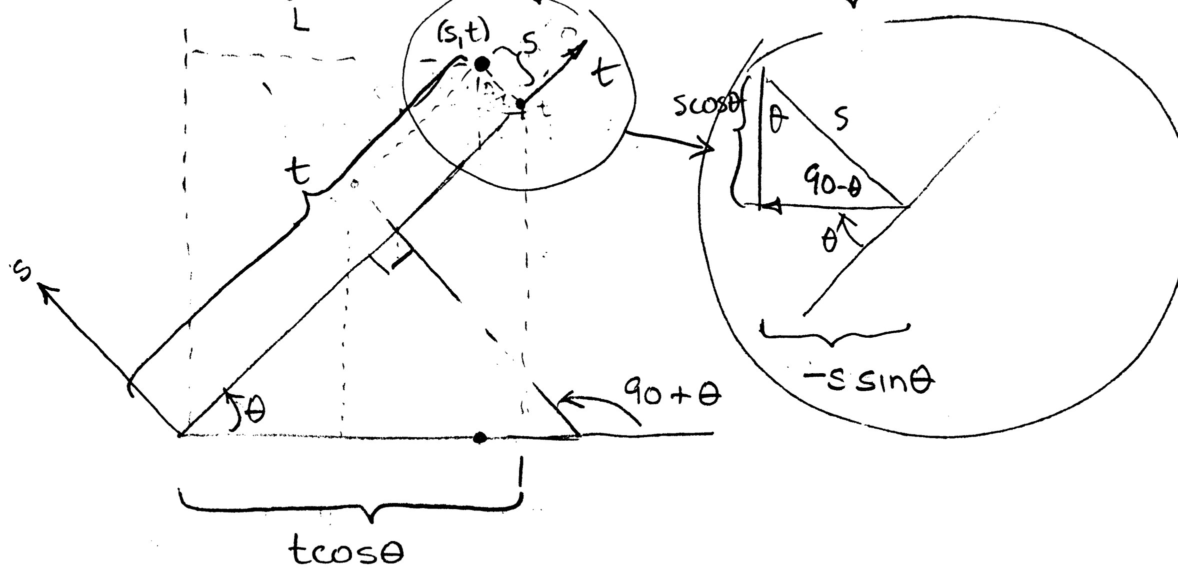
3.3 Projections



Equation of line is
$x\sin\theta - y\cos\theta + \rho = 0$
or, parametrically,
$x_0 = -\rho\sin\theta + s\cos\theta$
$y_0 = +\rho\cos\theta + s\sin\theta$

It is often more computationally efficient to use coded (reduced) data. One such form of coding is a projection (integral) onto a line

$$P_\theta(t) = \int_L b(t\cos\theta - s\sin\theta,\ t\sin\theta + s\cos\theta)\, ds$$



$$\therefore\quad x = t\cos\theta - s\cdot\sin\theta$$
$$y = t\sin\theta + s\cos\theta$$

(could be done by simple rotation)

if $\theta = 0$ (x-projection) $v(x) = P_0(t) = \int_L b(t,s)\,ds$

if $\theta = \frac{\pi}{2}$ (y-projection) $h(y) = P_{90^\circ}(t) = \int_L b(-s,t)\,ds$

One nice thing about projections is using them to compute moments.

Notice that $A = \iint b(x,y)\,dx\,dy$

$$= \int v(x)\,dx = \int h(y)\,dy$$

x-projection y-projection

for first moments:

$$\bar{x}A = \iiint x \; b(x,y)\,dx\,dy = \int x \; v(x)\,dx$$

since $\int b(x,y)\,dy = v(x)$

$$yA = \iint y \; b(x,y)\,dx\,dy = \int y \; h(y)\,dy$$

moment informatio is preserv

for second moments:

$$m_{02} = \iiint x^2\, b(x,y)\,dx\,dy = \int x^2\, v(x)\,dx$$

$$m_{20} = \iint y^2\, b(x,y)\,dx\,dy = \int y^2\, h(y)\,dy$$

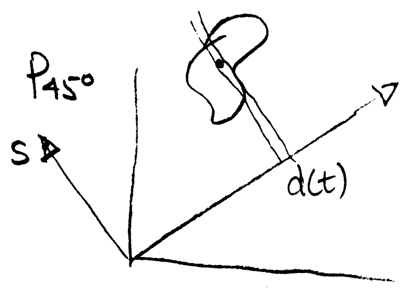
CANNOT compute cross-moment from x, y projections

$$m_{11} = \iint xy\; b(x,y)\,dx\,dy$$

moments do not give enough information. We need another moment:

Consider $d(t) = \int b\left(\frac{t-s}{\sqrt{2}}, \frac{t+s}{\sqrt{2}}\right) ds$

a projection onto a diagonal, i.e. $P_{45°}$



these terms cancel

$$\iint xy\, b(x,y)\, dxdy = \iint\left(\tfrac{1}{2}x^2 + xy + \tfrac{1}{2}y^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2\right) b(x,y)dxdy$$

$$= \iint \underbrace{\tfrac{1}{2}(x+y)^2}_{t^2} b(x,y)dxdy - \tfrac{1}{2}\iint x^2 \underbrace{b(x,y)dxdy}_{\text{projection onto } x} - \tfrac{1}{2}\iint y^2 \ldots$$

$$-\tfrac{1}{2}\int x^2 v(x)\, dx \quad -\tfrac{1}{2}\int y^2 f\ldots$$

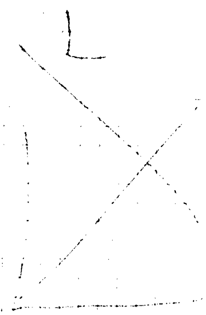
since $x = \dfrac{t-s}{\sqrt{2}}$

$y = \dfrac{t+s}{\sqrt{2}}$ along $45°$ diagonal.

$$x+y = \frac{2t}{\sqrt{2}} = \sqrt{2}t$$

$$(x+y)^2 = 2t^2$$

$$\therefore \iint xy\, b(x,y)\, dxdy = \int t^2\, b(x,y)\, ds dt - \tfrac{1}{2}\int x^2 v(x)\, dx - \tfrac{1}{2}\int y^2 h(y)\ldots$$



but the diagonal projection __IS__

diagonal projection $\underbrace{d(t)}_{\text{not a differential}} = \displaystyle\int_L b\left(\frac{t-s}{\sqrt{2}}, \frac{t+s}{\sqrt{2}}\right) ds.$

$$= \int t^2\, d(t)\, dt - \tfrac{1}{2}\int x^2 v(x)dx - \tfrac{1}{2}\int y^2 h(y)\, dy$$

Not attempting to do tomography but close.

micro texture properties

S = set of all pixels in R that are in a designated spatial relationship
(q.v. all pairs of pixels in R that are 4-neighbors)

$$P(g_1, g_2) = \frac{\#\{[(r_1,c_1),(r_2,c_2)] \in S \mid I(r_1,c_1) = g \text{ and } I(r_2,c_2) = g_2\}}{\# S}$$

texture second moment M
entropy E
correlation ρ
contrast C
homogeneity H

$$M = \sum_{g_1, g_2} P^2(g_1, g_2)$$

$$E = -\sum_{g_1, g_2} P(g_1, g_2) \log P(g_1, g_2)$$

$$\rho = \sum_{g_1, g_2} (g_1 - \mu)(g_2 - \mu) \frac{P(g_1, g_2)}{\sigma^2}$$

$$\mu = \frac{1}{2} \sum_{g_1} \sum_{g_2} g_1 P(g_1, g_2) + \sum_{g_1} \sum_{g_2} g_2 P(g_1, g_2)$$

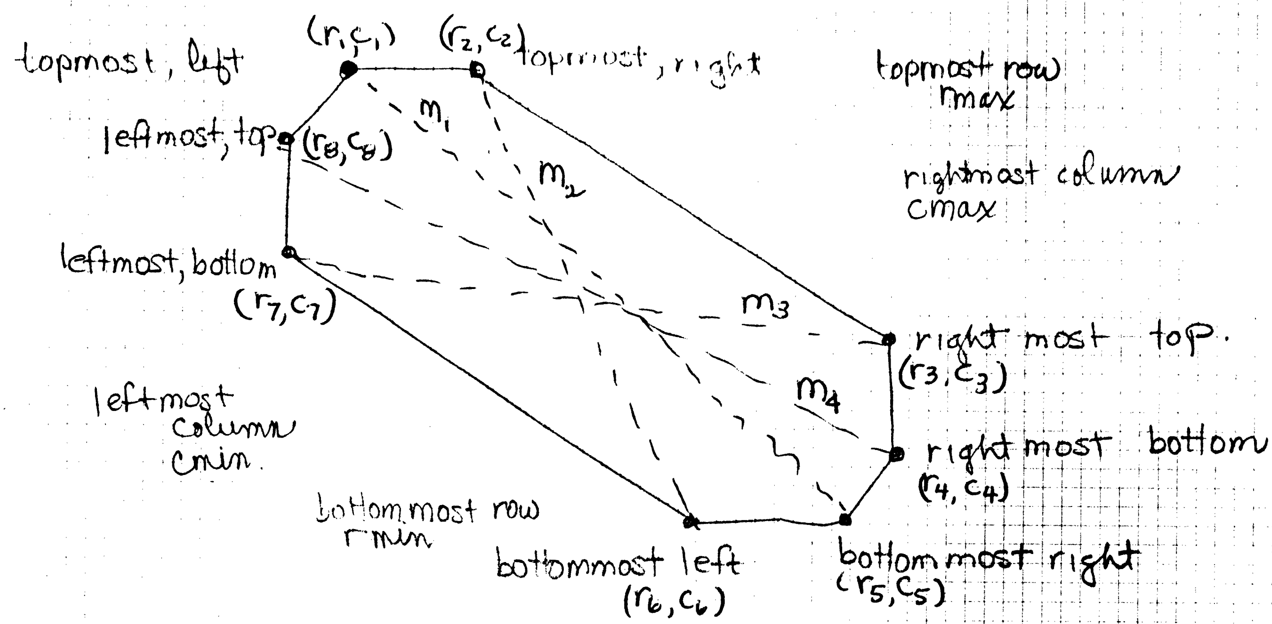
$$\sigma^2 = \frac{1}{2} \left[\sum \sum (g_1 - \mu)^2 P(g_1, g_2) + \sum_{g_1} \sum_{g_2} (g_2 - \mu)^2 P(g_1, g_2) \right]$$

$$C = \sum_{g_1} \sum_{g_2} |g_1 - g_2| P(g_1, g_2)$$

$$H = \sum_{g_1} \sum_{g_2} \frac{P(g_1, g_2)}{k + |g_1 - g_2|}$$

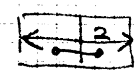
k is some small constant

3.2.1 Extremal points



extremal points occur in opposite pairs which define an axis

$$m_1 = \underbrace{\sqrt{(r_1-r_5)^2 + (c_1-c_5)^2}}_{\text{digital length}} + \underbrace{Q(\phi_1)}_{\text{correction}}$$



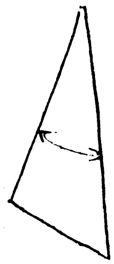
distance is 2 but length digital is only 1

do for all others

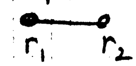
$$\text{where} \quad \phi_1 = \operatorname{Tan}^{-1}\left[\frac{r_1-r_5}{-(c_1-c_5)}\right]$$

m_1 mated with m_3
m_2 mated with m_4 } somewhat $\perp$

major axis is longest axis $= \max\{m_1, m_2, m_3, m_4\}$
minor is always mate of largest axis



$$\text{apex} = \max \underbrace{(r_1 + r_2)}_{\text{average of apex}}$$

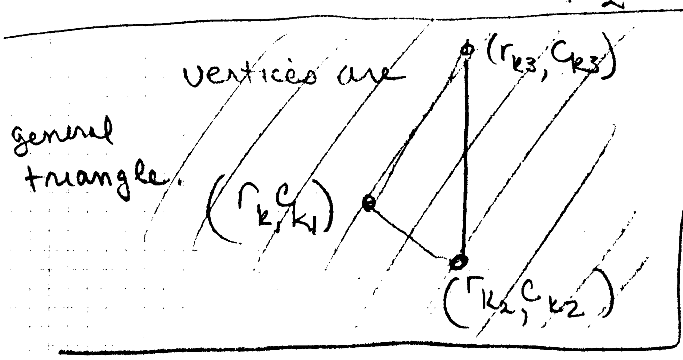


$$m_{ij} = \sqrt{(r_i - r_j)^2 + (c_i - c_j)^2} + 1.12 \qquad \text{for all axes.}$$

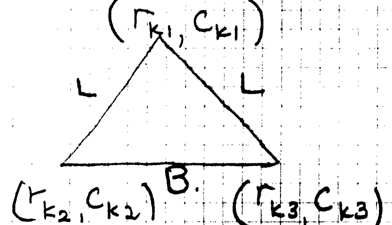
find k_1, k_2, k_3 (indices) to maximize $m_{k_1k_2} + m_{k_1k_3}$

vertices are

general triangle



for isosceles triangle



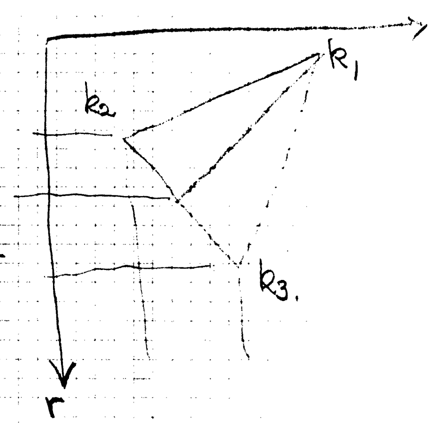
$$\Rightarrow \quad L = \frac{M_{k_1k_2} + m_{k_1k_2}}{2}$$

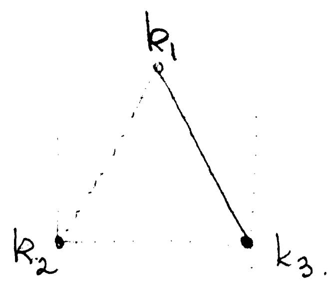
for isosceles triangle length L of long sides.

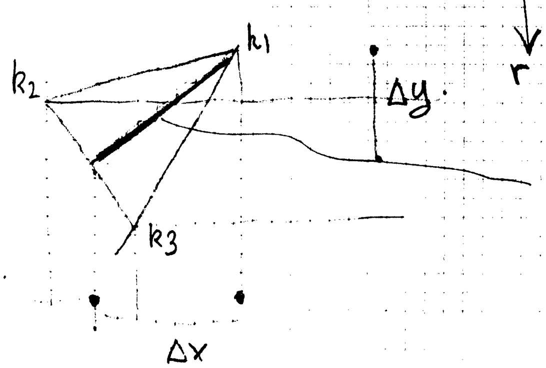
$$B = m_{k_2k_3}.$$ base

$$h = \sqrt{L^2 - \left(\frac{B}{2}\right)^2}$$

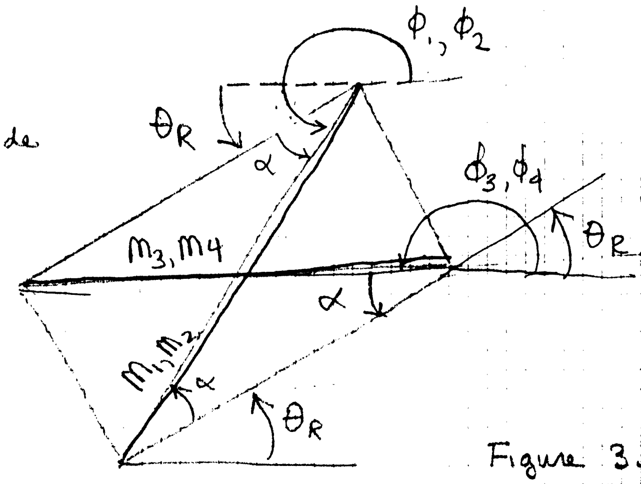
orientation $$\phi_h = \mathrm{Tan}^{-1} \frac{\frac{1}{2}(r_{k_2} + r_{k_3}) - r_{k_1}}{-\left[\frac{1}{2}(c_{k_2} + c_{k_3}) - c_{k_1}\right]}$$







α included angle between side and diagonal.



these figure shows well defined overlapping extremal points

Figure 3.8.

axes between extremal points (there are 8 extremal points, 4 axes)

m_1, m_2, m_3, m_4

two longest axes are diagonals of square or rectangle (they are mates).

$m_{(1)}$ = longest axes.

$m_{m(1)}$ = mate

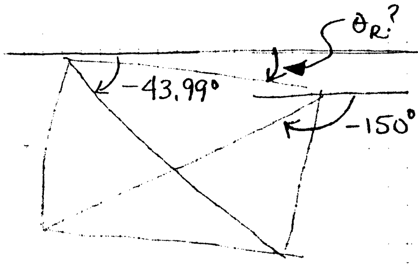
orientation of longest axis : $\phi_{(1)} = 180 + \theta_R + \alpha$ $\phi_2 = 180 + \theta_R + \alpha$

orientation of its mate $\phi_3 = 180 + \theta_R - \alpha$ $\phi_4 = 180 + \theta_R - \alpha$?

[measure angles at vertices!] $\theta_R = \dfrac{\phi_1 + \phi_{m1}}{2} - 180°$ [ϕ_1, ϕ_{m1} must be positive angles]

so add 360° to angle if that angle comes out negative

example :



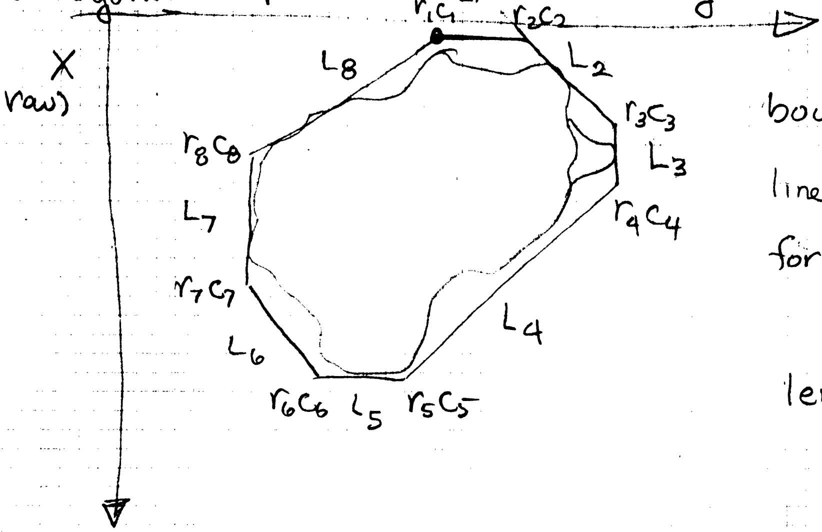
formulas don't agree

$$\theta_R = \frac{-43.99 - 150}{2} + 90 = -7.13°$$

this calculation does not agree with statement at top of p. 90 that all angles must be positive

$\theta_R \triangleq$ counterclockwise angle to the first side encountered from the horizontal axis and $0° \le \theta_R \le 90°$

octagonal shapes



bounding octagon for a region

line segment lengths are easily defined

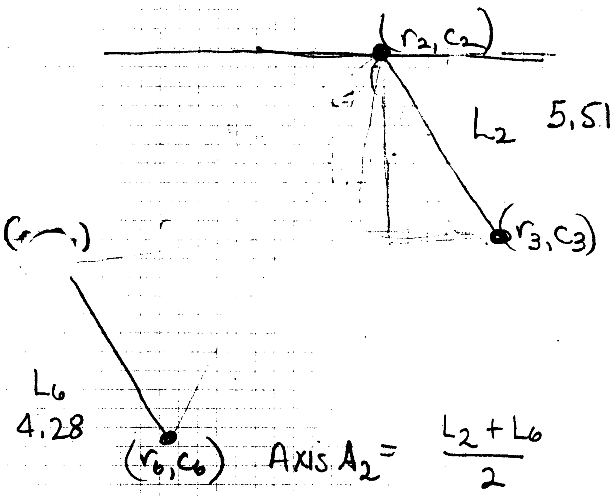
for example

$$L_1 = |c_1 - c_2| + 1$$

lengths of four axes.

$$A_1 = \frac{L_1 + L_5}{2}$$ always horizontal axis

$$A_2 = \frac{L_2 + L_6}{2}$$ ↗ axis angle varies



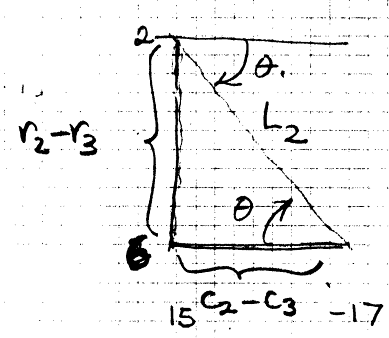
Axis $A_2 = \dfrac{L_2 + L_6}{2}$

average length

counter clockwise rotational angle.

$$\theta_2 = \frac{1}{2A_2}\left[L_2 \tan^{-1}\frac{r_2 - r_3}{-(c_2 - c_3)} + L_6 \tan^{-1}\frac{r_7 - r_6}{-(c_7 - c_6)}\right]$$

length weighted average



$r_2 - r_3 > 0$

$c_3 - c_2 > 0$

3.2.2. Spatial Moments

first order spatial moments

$$\bar{r} = \frac{1}{A} \sum_{(r,c) \in R} r$$

$$\bar{c} = \frac{1}{A} \sum_{(r,c) \in R} c$$

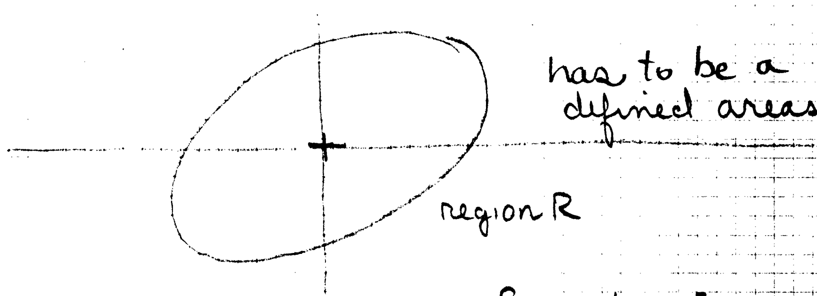
second order moments

2nd order row moment $$\mu_{rr} = \frac{1}{A} \sum_{(r,c) \in R} (r - \bar{r})^2$$

2nd order column moment $$\mu_{cc} = \frac{1}{A} \sum_{(r,c) \in R} (c - \bar{c})^2$$

2nd order mixed moment $$\mu_{rc} = \frac{1}{A} \sum_{(r,c) \in R} (r - \bar{r})(c - \bar{c})$$

very useful for elliptical shapes



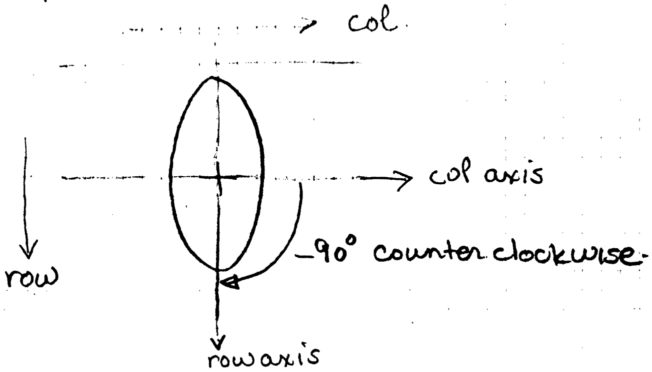
has to be a region with precisely defined areas.

$$R = \{ (r,c) \mid dr^2 + 2erc + fc^2 \leq 1 \}$$

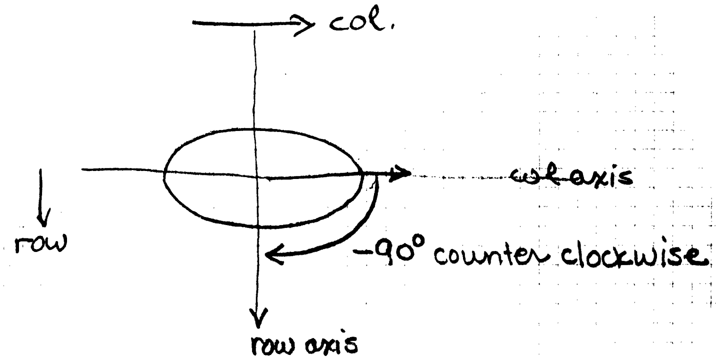
Appendix A

$$\begin{bmatrix} d & e \\ e & f \end{bmatrix} = \frac{1}{4(\mu_{rr}\mu_{cc} - \mu_{rc}^2)} \begin{bmatrix} \mu_{cc} & -\mu_{rc} \\ -\mu_{rc} & \mu_{rr} \end{bmatrix}$$

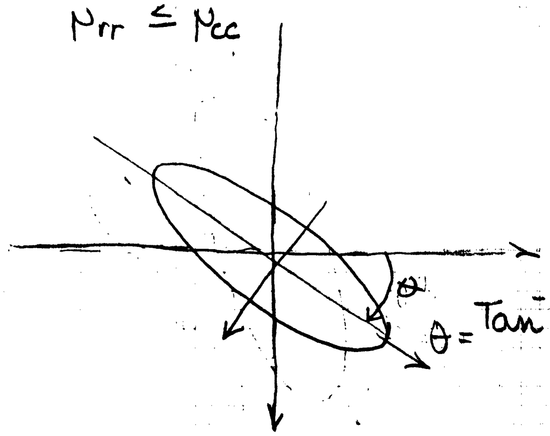
1. if $\mu_{rc} = 0$ $\quad \mu_{rr} > \mu_{cc}$



2. if $\mu_{rc} = 0$ $\quad \mu_{rr} \le \mu_{cc}$



3. if $\mu_{rc} \neq 0$ $\quad \mu_{rr} \le \mu_{cc}$

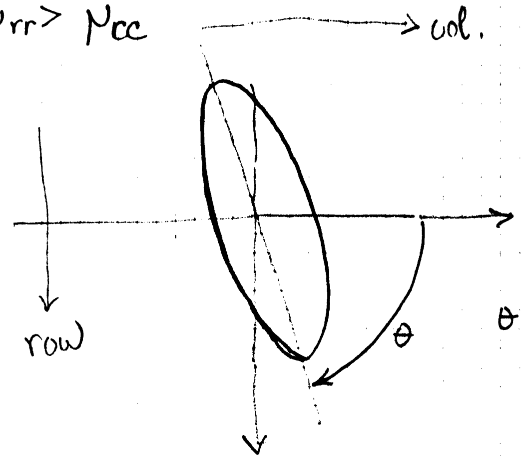


$$\theta = \mathrm{Tan}^{-1} \frac{-2\mu_{rc}}{\mu_{rr} - \mu_{cc} \pm \sqrt{(\mu_{rr} - \mu_{cc})^2 + 4\mu_{rc}^2}}$$

$$\text{length major axis} = \sqrt{8\left\{\mu_{rr} + \mu_{cc} + \sqrt{(\mu_{rr} - \mu_{cc})^2 + 4\mu_{rc}^2}\right\}}$$

$$\text{length minor axis} = \sqrt{8\left\{\mu_{rr} + \mu_{cc} - \sqrt{(\mu_{rr} - \mu_{cc})^2 + 4\mu_{rc}^2}\right\}}$$

4. $\mu_{rc} \neq 0 \quad \mu_{rr} > \mu_{cc}$



$$\theta = \operatorname{Tan}^{-1}\left\{ \frac{\sqrt{\mu_{cc} + \mu_{rr} + \sqrt{(\mu_{cc} - \mu_{rr})^2 + 4\mu_{rc}^2}}}{-2\mu_{rc}} \right.$$

length major axis $\sqrt{8\left\{ \mu_{rr} + \mu_{cc} + \sqrt{(\mu_{rr} - \mu_{cc})^2 + 4\mu_{rc}^2} \right\}}$

length minor axis $\sqrt{8\left\{ \mu_{rr} + \mu_{cc} + \sqrt{(\mu_{rr} - \mu_{cc})^2 + 4\mu_{cc}^2} \right.}$

3.2.3. Mixed Spatial Gray Level Moments.

2nd order, mixed gray-level spatial moments

$$\mu_{rg} = \frac{1}{A} \sum_{(r,c)\in R} (r - \bar{r})[I(r,c) - \mu]$$

$$\mu_{cg} = \frac{1}{A} \sum_{(r,c)\in R} (c - \bar{c})[I(r,c) - \mu]$$

least squares error of graylevel intensity plane and $I(r,c)$

$$\epsilon^2 = \sum_{(r,c)\in R} \left[\alpha(r - \bar{r}) + \beta(c - \bar{c}) + \gamma - I(r,c)\right]^2$$

taking partials of ϵ^2 w.r.t. α, β, γ and setting to zero gives

$$\begin{bmatrix} \sum\limits_{(r,c)\in R} (r-\bar{r})^2 & \sum\limits_{(r,c)\in R} (r-\bar{r})(c-\bar{c}) & \sum\limits_{(r,c)\in R} (r-\bar{r}) \\ \sum\limits_{(r,c)\in R} (r-\bar{r})(c-\bar{c}) & \sum\limits_{(r,c)\in R} (c-\bar{c})^2 & \sum\limits_{(r,c)\in R} (c-\bar{c}) \\ \sum\limits_{(r,c)\in R} (r-\bar{r}) & \sum\limits_{(r,c)\in R} (c-\bar{c}) & \sum\limits_{(r,c)\in R} 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \sum\limits_{(r,c)\in R} (r-\bar{r}) I(r,c) \\ \sum\limits_{(r,c)\in R} (c-\bar{c}) I(r,c) \\ \sum\limits_{(r,c)\in R} I(r,c) \end{bmatrix}$$

but $\sum\limits_{(r,c)} (r-\bar{r}) = 0$ and $\sum\limits_{(r,c)} (c-\bar{c}) = 0$

reducing to

moved to other side of equation

$$\begin{bmatrix} \sum\limits_{(r,c)\in R} (r-\bar{r})^2 & \sum\limits_{(r,c)\in R} (r-\bar{r})(c-\bar{c}) & 0 \\ \sum\limits_{(r,c)\in R} (r-\bar{r})(c-\bar{c}) & \sum\limits_{(r,c)\in R} (c-\bar{c})^2 & 0 \\ 0 & 0 & \sum\limits_{(r,c)\in R} 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \sum\limits_{(r,c)\in R} (r-\bar{r}) [I(r,c) - \gamma] \\ \sum\limits_{(r,c)\in R} (c-\bar{c}) [I(r,c) - \gamma] \\ \sum\limits_{(r,c)\in R} I(r,c) \end{bmatrix}$$

by inspection $\quad \gamma = \frac{1}{A} \sum\limits_{(r,c)} I(r,c) = \mu.$

recalling

$$\mu_{rr} = \frac{1}{A} \sum_{(r,c)} (r-\bar{r})^2$$

$$\mu_{rc} = \frac{1}{A} \sum_{(r,c)} (r-\bar{r})(c-\bar{c})$$

$$\mu_{cc} = \frac{1}{A} \sum_{(r,c)} (c-\bar{c})^2$$

$\Rightarrow$ We can re-write the above equations as

$$\begin{bmatrix} \mu_{rr} & \mu_{rc} \\ \mu_{rc} & \mu_{cc} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \mu_{rg} \\ \mu_{cg} \end{bmatrix}.$$

solve by Kramer's Rule

$$\alpha = \frac{\begin{vmatrix} \mu_{rg} & \mu_{rc} \\ \mu_{cg} & \mu_{cc} \end{vmatrix}}{\begin{vmatrix} \mu_{rr} & \mu_{rc} \\ \mu_{rc} & \mu_{cc} \end{vmatrix}} \qquad \beta = \frac{\begin{vmatrix} \mu_{rr} & \mu_{rg} \\ \mu_{rc} & \mu_{cg} \end{vmatrix}}{\begin{vmatrix} \mu_{rr} & \mu_{rc} \\ \mu_{rc} & \mu_{cc} \end{vmatrix}}$$

gray scale fit to region is

$$\hat{I}(r,c) = \alpha\,(r-\bar{r}) + \beta(c-\bar{c}) + \mu \qquad (r,c) \in R$$

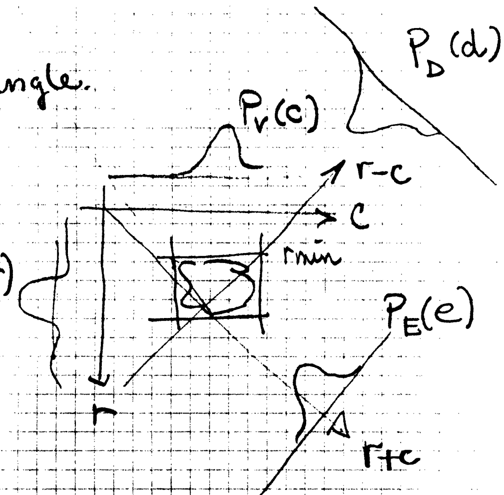
3.3. Signature Properties

properties obtainable from vertical, horizontal & diagonal projections

- area
- centroid
- second moments
- bounding rectangles

signature analysis

- orientation and position of a rectangle.
- position of a circle.



recall definitions of projections

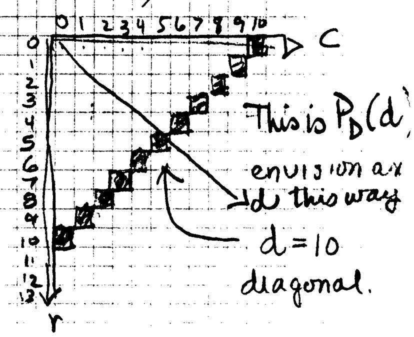
$$P_V(c) = \#\{r \mid (r,c) \in R\}$$

$$P_H(r) = \#\{c \mid (r,c) \in R\}$$

diagonal projections.

$$P_D(d) = \#\{(r,c) \in R \mid r+c = d\}$$

$$P_E(e) = \#\{(r,c) \in R \mid r-c = e\}$$



Area can be obtained from any projection

$$A = \sum_{(r,c) \in R} 1 = \sum_r \sum_{\{c \mid (r,c) \in R\}} 1 = \sum_r P_H(r)$$

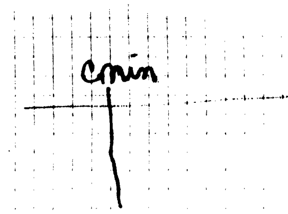
top row of bounding rectangle.

$$rmin = \min\{r \mid (r,c) \in R\}$$
$$= \min\{r \mid P_H(r) \neq \phi\}$$

bottom row

$$rmax = \max\{r \mid (r,c) \in R\}$$
$$= \max\{r \mid P_H(r) \neq 0\}.$$

left most column



$$c_{min} = \min\{c \mid (r,c) \in R\}$$
$$= \min\{c \mid P_V(c) \neq 0\}$$

right most column

$$c_{max} = \max\{c \mid (r,c) \in R\}$$
$$= \max\{c \mid P_V(c) \neq 0\}$$

row centroid

is is basically weighted by population

this is $P_H(r)$

$$\bar{r} \triangleq \frac{1}{A} \sum_{(r,c)\in R} r = \frac{1}{A} \sum_{r} \sum_{\{c \mid (r,c)\in R\}} r = \frac{1}{A} \sum_{r} r \sum_{\{c \mid (r,c)\in R\}} 1 = \frac{1}{A} \sum_{r} r P_H(r)$$

for a given r

similarily,

this is $P_V(c)$

$$\bar{c} = \frac{1}{A} \sum_{(r,c)\in R} c = \frac{1}{A} \sum_{c} \sum_{\{r \mid (r,c)\in R\}} c = \frac{1}{A} \sum_{c} c \sum_{\{c \mid (r,c)\in R\}} 1 = \frac{1}{A} \sum_{c} c P_V(c)$$

Diagonal centroids

$$\bar{d} = \frac{1}{A} \sum_{d} d P_D(d) \qquad d = r + c$$

$$\bar{e} = \frac{1}{A} \sum_{e} e P_E(e) \qquad e = r - c$$

Relationship between row & column centroids

moved d inside

since $r+c=d$

$$\bar{d} = \frac{1}{A}\sum_d d \overbrace{\sum_{\{(r,c)\in R \mid r+c=d\}} 1}^{P_D(d)} = \frac{1}{A}\sum_d \sum_{\{(r,c)\in R \mid r+c=d\}} (r+c)$$

$$= \frac{1}{A}\sum_d \sum_{\{(r,c)\in R \mid r+c=d\}} r + \frac{1}{A}\sum_d \sum_{\{(r,c)\in R \mid r+c=d\}} c$$

$$= \frac{1}{A}\sum_{(r,c)\in R} r + \frac{1}{A}\sum_{(r,c)\in R} c$$

Note this works like partial integrals

$$\therefore \quad \bar{d} = \bar{r} + \bar{c}$$

The second row moment μ_{rr} can be obtained from P_H

$$\mu_{rr} = \frac{1}{A}\sum_{(r,c)\in R} (r-\bar{r})^2$$

by definition

but this is $P_H(r)$!

$$= \frac{1}{A}\underbrace{\sum_r \sum_{\{c \mid (r,c)\in R\}}}_{\text{separating}} (r-\bar{r})^2 = \frac{1}{A}\sum_r (r-\bar{r})^2 \sum_{\{c \mid (r,c)\in R\}} 1$$

not dependent on c so move outside

$$= \frac{1}{A}\sum (r-\bar{r})^2 \, P_H(r)$$

The second column moment μ_{cc}

$$\mu_{cc} = \frac{1}{A}\sum_{(r,c)\in R} (c-\bar{c})^2 = \frac{1}{A}\sum_c \sum_{\{r \mid (r,c)\in R\}} (c-\bar{c})^2$$

$$= \frac{1}{A}\sum_c (c-\bar{c})^2 \sum_{\{r \mid (r,c)\in R\}} 1 = \frac{1}{A}\sum_c (c-\bar{c})^2 \, P_V(c)$$

The second diagonal moment μ_{dd}

$$\mu_{dd} = \frac{1}{A} \sum_{d} (d-\bar{d})^2 P_D(d)$$ can be obtained in the same way.

move inside!

$$\mu_{dd} = \frac{1}{A} \sum_{d} \sum_{\{(r,c)\in R \mid r+c=d\}} (r+c-\bar{r}-\bar{c})^2$$

$$= \frac{1}{A} \sum_{(r,c)\in R} [(r-\bar{r})+(c-\bar{c})]^2$$ expanding and go back to a single sum, just defin

$$= \frac{1}{A} \sum_{(r,c)\in R} [(r-\bar{r})^2 + 2(r-\bar{r})(c-\bar{c}) + (c-\bar{c})^2]$$

$$= \mu_{rr} + 2\mu_{rc} + \mu_{cc}$$ using definitions

As an aside, this says that

$$\mu_{rc} = \frac{\mu_{dd} - \mu_{rr} - \mu_{cc}}{2}$$ the 2nd mixed component can be computed from the 2nd diagonal component.

The other diagonal component:

rearranging

$$\mu_{ee} = \frac{1}{A} \sum_{e} \sum_{\{(r,c)\in R \mid r-c=e\}} [(r-c)-(\bar{r}-\bar{c})]^2 = \frac{1}{A} \sum_{(r,c)\in R} [(r-\bar{r})-(c-\bar{c})]^2$$

expanding

$$= \frac{1}{A} \sum_{(r,c)\in R} [(r-\bar{r})^2 - 2(r-\bar{r})(c-\bar{c}) + (c-\bar{c})^2]$$

$$\mu_{ee} = \mu_{rr} - 2\mu_{rc} + \mu_{cc}$$

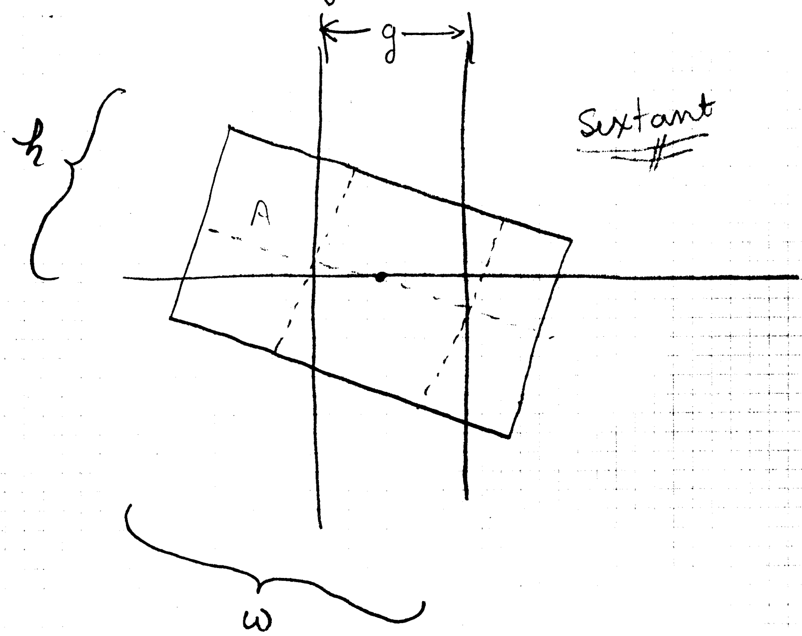
and, similarily, $\mu_{rc} = \dfrac{\mu_{rr} + \mu_{cc} - \mu_{ee}}{2}$

Combining the expressions for μ_{rc}

$$2\mu_{rc} = \frac{\mu_{dd} - \mu_{ee}}{2}$$

$$\mu_{rc} = \frac{\mu_{dd} - \mu_{ee}}{4}$$

3.3.1 Using Signature Analysis to determine the Center and Orientation of a rectangle.



6 rectangular regions
projection index image

Assumptions
- corners are in extreme sextants.
- h, w known
- one object (masking) using a bounding box

2.4. signature analysis

take one or more projections of a binary image, segmenting each projection, and taking property measurements of each projection segment.

(first used in character recognition, can be done in real time)

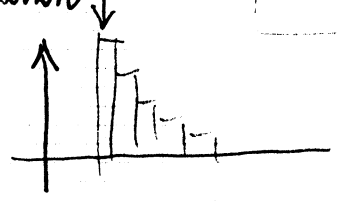
each pixel of resulting image has value 0 if corresponding pixel in binary image is zero,
Signature is histogram of non-zero pixels of resulting masked image.

projection index image. mask with given binary image

1	2	3	4
1	2	3	4
1	2	3	4
1	2	3	4
1	2	3	4
1	2	3	4
1	2	3	4

i.e. you weight it somehow.
Actually the pixels take on the values of the mask image.

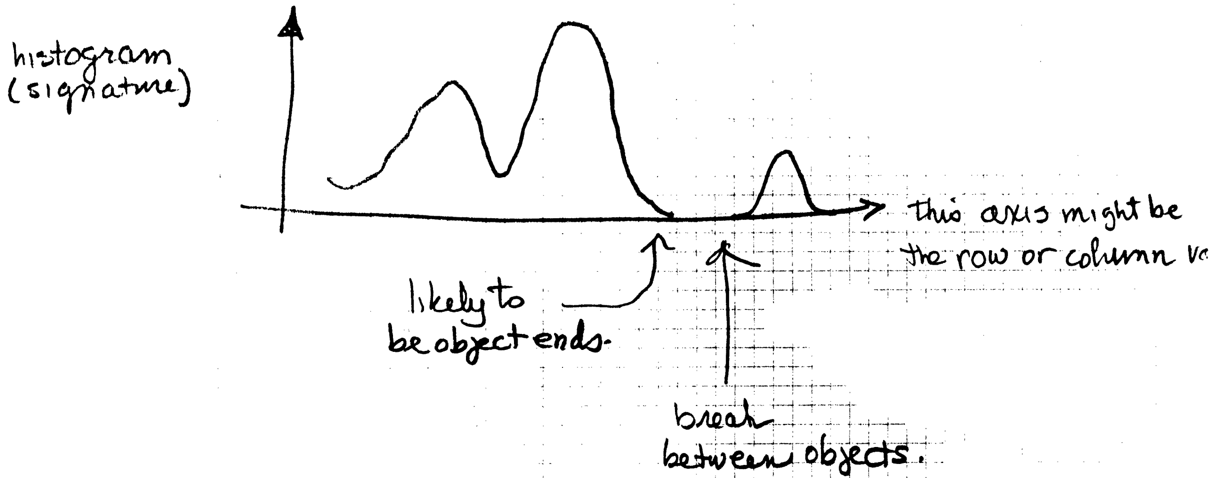
population of each column



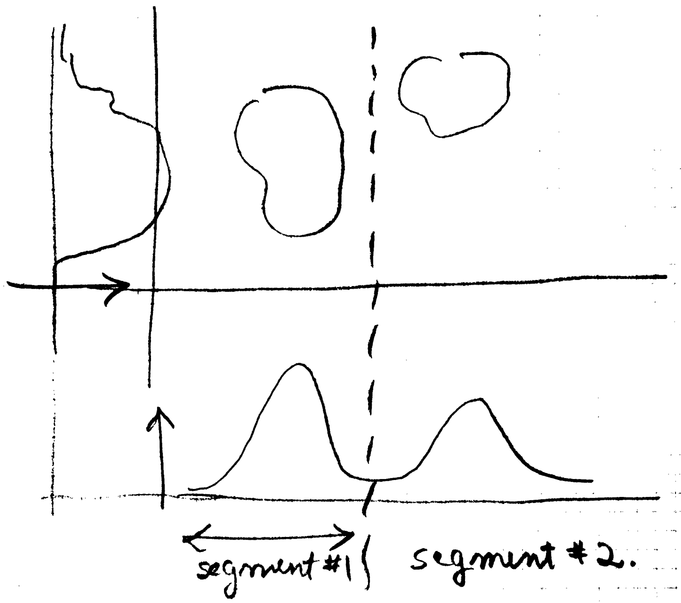
0 if zero in binary image
c if one in binary image.

signature is histogram of non zero values.

observations



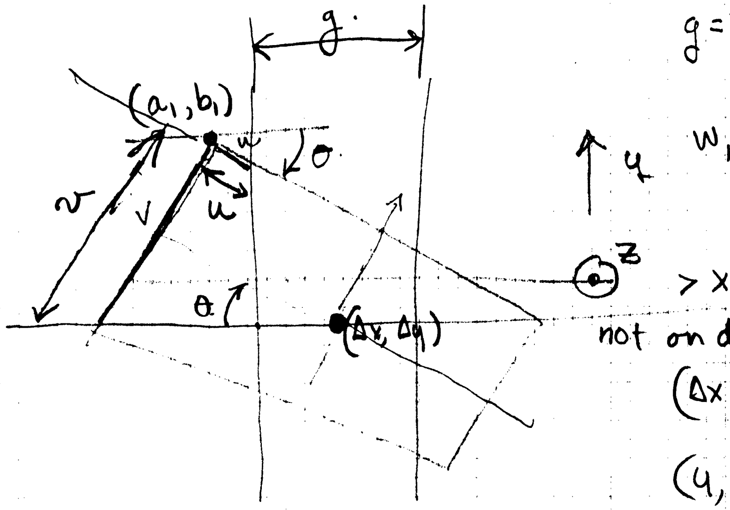
recursive segmentation procedure.



threshold and re-segment.

final phase — feature measurements of each segment
(these are the real signature).

- sum of all projection values — area
- weighted sum (centroid)
- #'s and height of peaks and valleys.
- functional fits ⟶ feature vector
- normalized length ⟶ histogram scaling.



g = horizontal spacing between sextant dividing lines.

w, h – a priori known width and height of rectangle.

not on diagonals – an accident

$(\Delta x, \Delta y)$ coordinates of the center of the rectangle.

(u, v) line segment lengths in upper left sextant

1. do projections to get six areas:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Rotation θ about z.

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \underbrace{\begin{pmatrix} -\frac{w}{2} \\ \frac{h}{2} \end{pmatrix}}_{\text{in rectangle frame}} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

actually rotating $\circlearrowleft \theta$.

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \overbrace{\begin{pmatrix} a_1 + \frac{w}{2}\cos\theta - \frac{h}{2}\sin\theta \\ b_1 - \frac{w}{2}\sin\theta - \frac{h}{2}\cos\theta \end{pmatrix}}^{\text{rotate} \qquad \text{move origin}}$$

let $a_1 = -\frac{g}{2} - u\cos\theta$

$b_1 = v\cos\theta$.

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -\frac{g}{2} - u\cos\theta + \frac{w}{2}\cos\theta - \frac{h}{2}\sin\theta \\ v\cos\theta - \frac{w}{2}\sin\theta - \frac{h}{2}\cos\theta \end{pmatrix}.$$

determine u, v in terms of A, B, C, D, E, F