EEAP 210

ELECTROMAGNETIC FIELDS

EXAMINATION NO 1 – SOLUTION SET

(1)

$\underline{A} = 3\underline{a_x} + 2\underline{a_y} - \underline{a_z}$

$\underline{B} = 3\underline{a_x} - 4\underline{a_y} - 5\underline{a_z}$

$\underline{C} = \underline{a_x} - \underline{a_y} + \underline{a_z}$

(a) $\underline{A} + \underline{B} = 6\underline{a_x} - 2\underline{a_y} - 6\underline{a_z}$

(b) $\underline{A} \cdot \underline{B} = 9 - 8 + 5 = 6$

(c) $\underline{A} \cdot \underline{B} \cdot \underline{C}$ is nonsense

(d)

$$\underline{A} \times \underline{B} = \begin{bmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ 3 & 2 & -1 \\ 3 & -4 & -5 \end{bmatrix} = (-10\underline{a_x} - 3\underline{a_y} - 12\underline{a_z}) - (4\underline{a_x} - 15\underline{a_y} + 6\underline{a_z})$$

$$= -14\underline{a_x} + 12\underline{a_y} - 18\underline{a_z}$$

(e) let θ be the angle between $\underline{A}$ and $\underline{B}$

$$\underline{A} \cdot \underline{B} = |\underline{A}||\underline{B}| \cos\theta$$

$$\cos\theta = \frac{\underline{A} \cdot \underline{B}}{|A||B|}$$ where $|\underline{A}| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}$

$|\underline{B}| = \sqrt{3^2 + (-4)^2 + (-5)^2} = \sqrt{50}$

$$= \frac{6}{\sqrt{50}\sqrt{14}} \approx 0.227$$

$\theta \approx 76.89$ degrees

alternatively,

$$|\underline{A} \times \underline{B}| = |\underline{A}||\underline{B}| \sin\theta$$

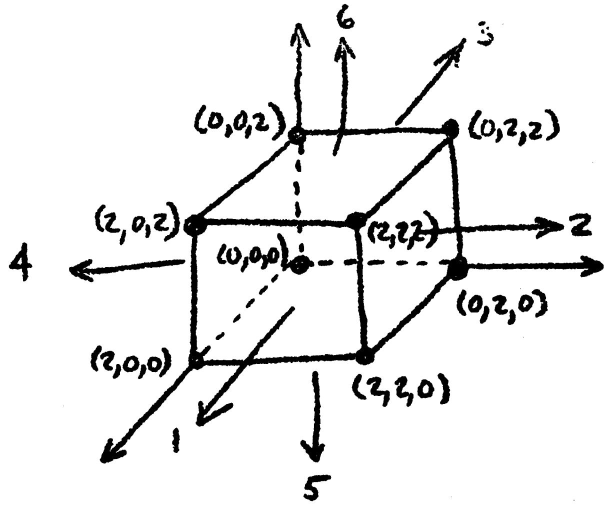
From above $|\underline{A} \times \underline{B}| = |-14\underline{a_x} + 12\underline{a_y} - 18\underline{a_z}|$

$$= \sqrt{196 + 144 + 324} = \sqrt{664}$$

$$\sin\theta = \frac{|\underline{A} \times \underline{B}|}{|\underline{A}||\underline{B}|} = \frac{\sqrt{664}}{\sqrt{14}\sqrt{50}} \approx 0.974$$

$\theta \approx 76.89$ degrees.

②



(a) Evaluate $\oint \underline{A} \cdot \underline{ds}$ by direct evaluation

side		
1	$\underline{ds} = \underline{a_x}\, dy\, dz$	$xy^2 \underline{a_x} \cdot \underline{a_x}\, dy\, dz = xy^2\, dy\, dz$ for $x = 2$
2	$\underline{ds} = \underline{a_y}\, dx\, dz$	$xy^2 \underline{a_x} \cdot \underline{a_y}\, dx\, dz \longrightarrow 0$
3	$\underline{ds} = -\underline{a_x}\, dy\, dz$	$xy^2 \underline{a_x} \cdot -\underline{a_x}\, dy\, dz = -xy^2\, dy\, dz$ for $x = 0$
4	$\underline{ds} = -\underline{a_y}\, dx\, dz$	$xy^2 \underline{a_x} \cdot -\underline{a_y}\, dx\, dz \longrightarrow 0$
5	$\underline{ds} = -\underline{a_z}\, dx\, dy$	$xy^2 \underline{a_x} \cdot -\underline{a_z}\, dx\, dy \longrightarrow 0$
6	$\underline{ds} = \underline{a_z}\, dx\, dy$	$xy^2 \underline{a_x} \cdot \underline{a_z}\, dx\, dy \longrightarrow 0$

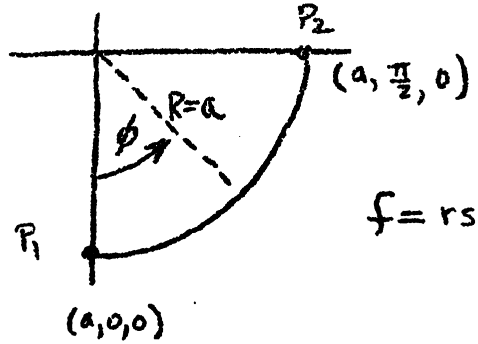
Because of the unidirectional field ($\underline{a_x}$ direction only) four of the six sides have no contribution to the integral. Only sides 1 and 3 contribute and the contribution from 3 goes to zero because $x=0$ on that side; thus, $\underline{A} = 0$ there.

$$\therefore \oint_S \underline{A} \cdot \underline{ds} = \int_0^2 \int_0^2 2y^2\, dy\, dz = 2\left(\frac{y^3}{3}\right)\Big|_0^2 (2-0) = \frac{4}{3}\left(y^3\Big|_0^2\right) = \frac{32}{3}$$

(b) $\oint_S \underline{A} \cdot \underline{ds}$ is the NET (or total) flux through the volume bounded by S

(c) $\oint_S \underline{A} \cdot \underline{ds} > 0$ indicates that there is a source of flux within V (bounded by S)

③



$f = r\sin\phi$ in cyclindrical coordinates

to evaluate $\int_{P_1}^{P_2} \underline{\text{grad}}\, f \cdot \underline{dl}$

first evaluate $\underline{\nabla} f = \underline{a_r} \frac{\partial}{\partial r}(r\sin\phi) + \underline{a_\phi} \frac{1}{r}\frac{\partial}{\partial\phi}(r\sin\phi) + \underline{a_z} \frac{\partial(r\sin\phi)}{\partial z}$

$$= \underline{a_r} \sin\phi + \underline{a_\phi} \frac{1}{r} \cdot r\cos\phi + 0$$

$$= \underline{a_r} \sin\phi + \underline{a_\phi} \cos\phi$$

$$\underline{dl} = \underline{a_r}\, dr + \underline{a_\phi}\, r\, d\phi + \underline{a_z}\, dz$$

many people forgot the r in the $\underline{a_\phi}$ term

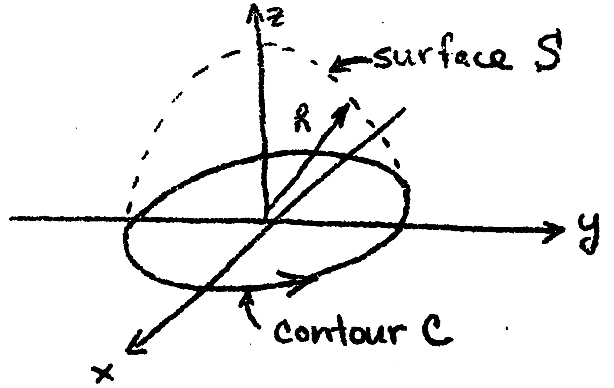
$$\underline{\nabla f} \cdot \underline{dl} = (\underline{a_r} \sin\phi + \underline{a_\phi}\cos\phi) \cdot (\underline{a_r}\, dr + \underline{a_\phi}\, r\, d\phi + \underline{a_z}\, dz)$$

$$= \sin\phi\, dr + r\cos\phi\, d\phi$$

$$\int_{P_1}^{P_2} \underline{\nabla f} \cdot \underline{dl} = \underbrace{\int_a^a \sin\phi\, dr}_{\substack{\text{r does not change} \\ \text{along the contour} \\ \text{so this integral goes to zero.}}} + a\int_0^{\frac{\pi}{2}} \cos\phi\, d\phi = a\int_0^{\frac{\pi}{2}} \cos\phi\, d\phi = a\sin\phi \Big|_0^{\frac{\pi}{2}} = a$$

$$\int_{P_1}^{P_2} \nabla \underline{f} \cdot \underline{dl} = a$$

4



(a) $\oint_C \underline{A} \cdot \underline{dl}$

$$\left.\begin{aligned} \underline{dl} &= r\,d\phi\,\underline{a_\phi} + dr\,\underline{a_r} + dz\,\underline{a_z} \\ \underline{A} &= r\,\underline{a_\phi} - z\,\underline{a_z} \end{aligned}\right\} \text{in cyclindrical coordinates}$$

$$\begin{aligned} \underline{A} \cdot \underline{dl} &= (r\underline{a_\phi} - z\underline{a_z}) \cdot (r\,d\phi\,\underline{a_\phi} + dr\,\underline{a_r} + dz\,\underline{a_z}) \\ &= r^2 d\phi - z\,dz \end{aligned}$$

$$\oint_C \underline{A} \cdot \underline{dl} = \int_0^{2\pi} r^2 d\phi - \underbrace{\int_0^0 z\,dz}_{} = 2\pi R^2 \quad (\text{as } r = R)$$

this integral has these limits and goes to zero as it did because C is in the xy (z=0) plane.

(b) by Stokes' Theorem <u>ANY</u> open surface attached to the contour C has the same integral of the curl as the line integral around C, i.e.

$$\int_S (\underline{\nabla} \times \underline{A}) \cdot \underline{ds} = \oint_C \underline{A} \cdot \underline{dl} = 2\pi R^2$$

where S is as shown in the drawing above.

EEAP 210 ELECTROMAGNETIC FIELD THEORY

FEBRUARY 20, 1983

EXAM NO. 2.

NAME : ______________________________

Instructions:

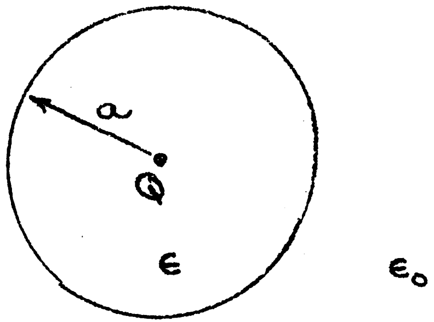
1. Do ALL PROBLEMS (THERE ARE THREE (3))
2. WRITE NEATLY, PREFERABLY IN PENCIL
3. READ THE PROBLEM CAREFULLY
4. THINK !
5. Show all work — state your assumptions.
6. Staple your formula sheet to the exam – it will be returned to you.

GRADING:

PROBLEM 1		3 points possible
2		3 points possible
3		5 points possible
TOTAL		11 points possible

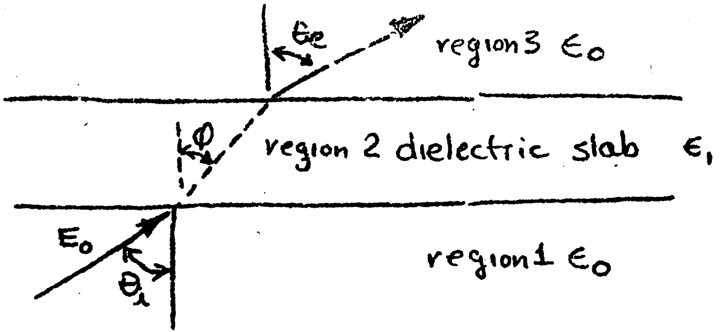
ADVISORY GRADE

1.



A point charge Q is placed at the center of a dielectric sphere of permittivity ϵ and radius a. Assume the sphere is in free space (ϵ_0). Find $\underline{D}$ and $\underline{E}$ everywhere (use Gauss' Law).

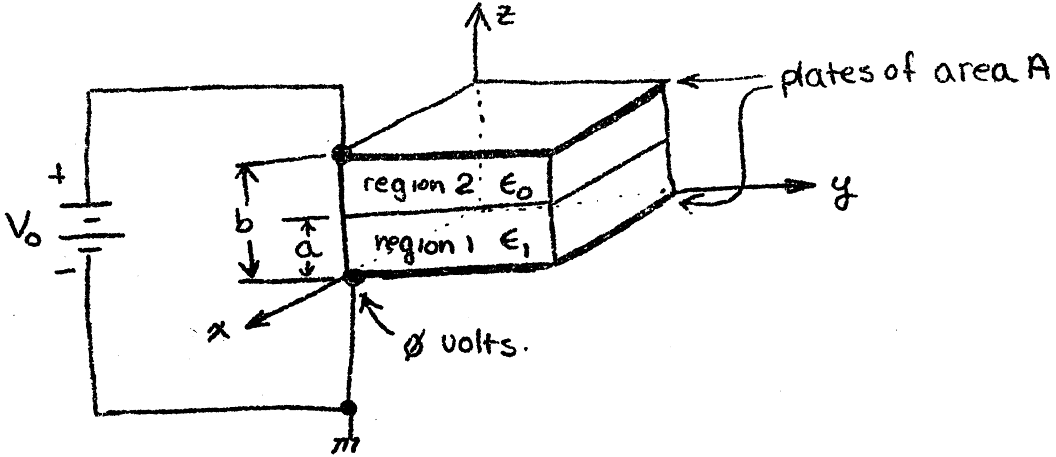
2.



A dielectric slab is immersed in a uniform electric field in free space. If the entrance angle θ_i for the electric field is known in region 1, find the field components exiting the slab into region 3.

Hint: Assume an angle θ_e for the exit angle.

3. Consider the parallel plate capacitor with two layers of dielectric as shown below



(a) Calculate the $\underline{E}$ field and the electrostatic potential Φ everywhere. State all assumptions.

3 (continued)

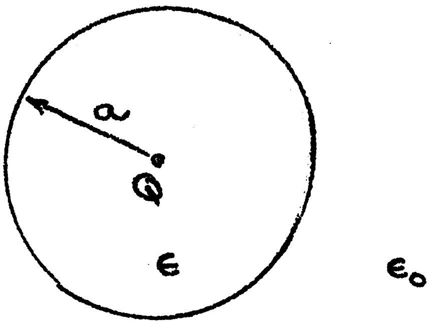
(b) Suppose one can inject free charge into the interface between the dielectrics (at $z=a$). How much charge must be injected to make the field in region 2 go to zero? Of what sign must the injected charge be?

(c) What is the capacitance of this parallel-plate capacitor?

Exam #2

$\int \sin\theta = -\cos\theta \Big|$

1.



A point charge Q is placed at the center of a dielectric sphere of permittivity ϵ and radius a. Assume the sphere is in free space (ϵ_0). Find $\underline{D}$ and $\underline{E}$ everywhere (use Gauss' Law).

Gauss' Law $\qquad \int \underline{D} \cdot d\underline{s} = \int \rho \, dO = Q.$

$$d\underline{S} = r^2 \sin\theta \, d\theta \, d\phi \, \underline{a_r}$$

$\underline{D} = \epsilon E_r \underline{a_r}$ all other components being zero.

$$\int_0^{2\pi}\int_0^{\pi} \epsilon E_r r^2 \sin\theta \, d\theta \, d\phi = \epsilon E_r r^2 \, 2\pi \int_0^{\pi} \sin\theta \, d\theta$$

$$= 2\pi \epsilon r^2 E_r \left[-\cos\theta \Big|_0^{\pi} \right]$$

$$= 2\pi \epsilon r^2 E_r \left[-(-1) + (1) \right]$$

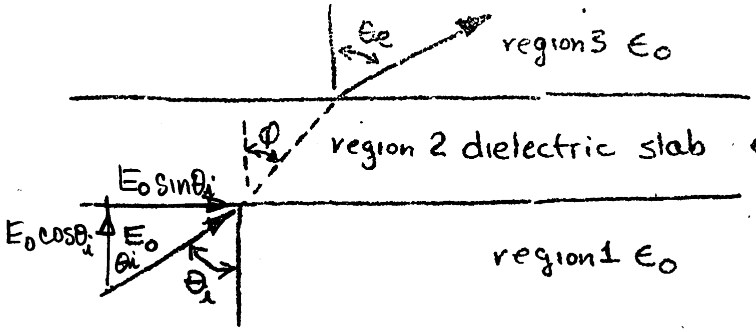
$$= 4\pi \epsilon r^2 E_r = Q.$$

$$\boxed{\begin{aligned} E_r &= \frac{Q}{4\pi\epsilon r^2} \qquad r \leq a \\ \underline{D_r} &= \epsilon E_r = \frac{Q}{4\pi r^2} \end{aligned}}$$

no charge at surface so.

$$\left.\begin{aligned} \underline{D}r &= \frac{Q}{4\pi r^2} \\ \underline{E}r &= \frac{Q}{4\pi\epsilon_0 r^2} \end{aligned}\right\} \quad r > a \qquad \therefore D \text{ is not discontinuous}$$

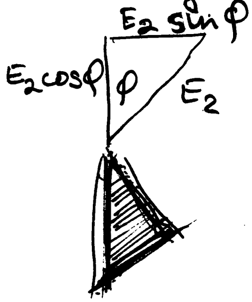
2.



A dielectric slab is immersed in a uniform electric field in free space. If the entrance angle θ_i for the electric field is known in region 1, find the field components exiting the slab into region 3.

Hint: Assume an angle θ_e for the exit angle.

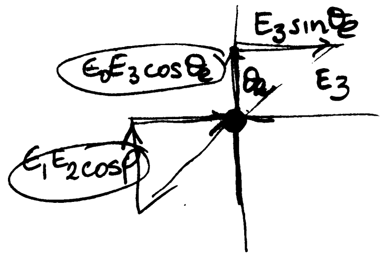
inside slab



$(D_{\text{region }1})_{\text{normal}}$ $(D_{\text{region }2})_{\text{normal}}$

$$\epsilon_0 E_0 \cos\theta_i = \epsilon_1 E_2 \cos\phi$$

$$E_2 \sin\phi = E_0 \sin\theta_i$$



$$E_3 \sin\theta_e = E_2 \sin\phi = E_0 \sin\theta_i$$

$$\epsilon_1 E_2 \cos\phi = \epsilon_0 E_3 \cos\theta_e$$

$$\therefore \quad \epsilon_0 E_0 \cos\theta_i = \epsilon_0 E_3 \cos\theta_e$$

tangential: $E_0 \sin\theta_i = E_3 \sin\theta_e$

normal: $E_0 \cos\theta_i = E_3 \cos\theta_e$

also note

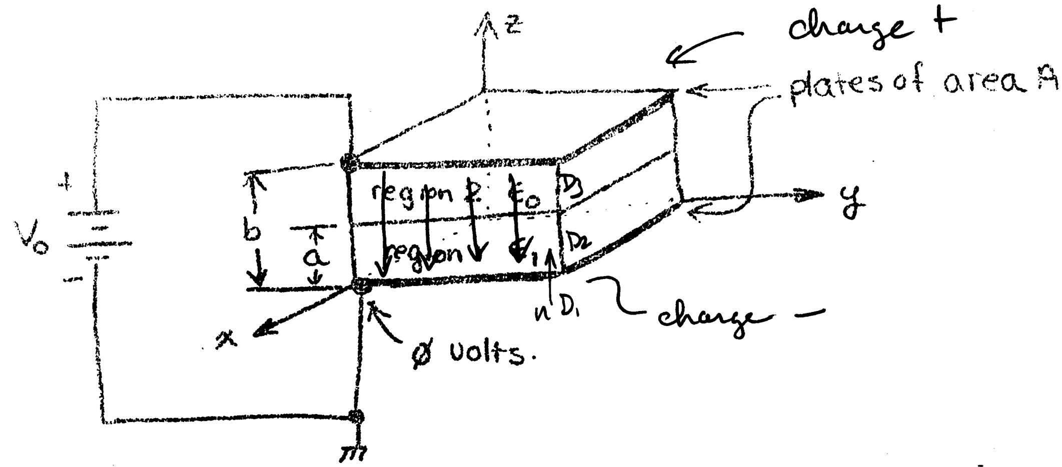
and $E_0 \sin\theta_i = E_0 \sin\phi = E_0 \sin\theta_e$

$\epsilon_0 E_0 \cos\theta_i = \epsilon_1 E_0 \cos\phi = \epsilon_0 E_0 \cos\theta_e$

or $\dfrac{\tan\theta_i}{\epsilon_0} = \dfrac{\tan\phi}{\epsilon_1} = \dfrac{\tan\theta_e}{\epsilon_0}$

0.

3. Consider the parallel plate capacitor with two layers of dielectric as shown below.



(a) Calculate the $\underline{E}$ field and the electrostatic potential Φ everywhere. State all assumptions.

$$D_{2n} = \frac{Q}{A} = \epsilon_1 E.$$

$$E = \frac{Q}{\epsilon_1 A} = -c_{11}$$

$$c_{11} = -\frac{Q}{\epsilon_1 A}.$$

in capacitor $\nabla^2\Phi = 0$

$$\frac{d^2\Phi}{dz^2} = 0$$

$$\frac{d\Phi}{dz} = c_1$$

$$\Phi(z) = c_1 z + c_2.$$

at $z=0$ $\Phi(z=0) = c_2 = \phi$

$$\underline{E} = -\nabla\Phi = -\frac{d}{dz}(c_1 z) = -c_1$$

$$\underline{D} = \epsilon_1 \underline{E} = -\epsilon_1 c_1$$

$\underline{n}\cdot(\underline{D_2}-\underline{D_1}) = \rho_s$ $\quad \underline{a_z}\cdot(-\epsilon_1 c_1 \underline{a_z} - \phi) = \rho_s$

$$-\epsilon_1 c_1 = \rho_s$$

$$c_1 = -\frac{\rho_s}{\epsilon_1} \Rightarrow$$

$0 < z < a.$

$$\boxed{\begin{aligned}\Phi(z) &= -\frac{\rho_s z}{\epsilon_1}\\ E_2(z) &= \frac{\rho_s}{\epsilon_1}\\ D_2(z) &= -\epsilon_1\left(-\frac{\rho_s}{\epsilon_1}\right) = \rho_s\end{aligned}}$$

in region 2 $\quad \Phi(z) = c_3 z + c_4$

at $z=a$ $\quad \Phi(z=a) = c_3 a + c_4 = -\frac{\rho_s a}{\epsilon_1}$ $\quad \underline{n}\cdot(\underline{D_3}-\underline{D_2}) = 0$

$$\underline{E} = -\frac{d}{dz}\Phi = -c_3$$

$$\underline{D} = \epsilon_0 \underline{E} = -\epsilon_0 c_3$$

$$\underline{a_z}\cdot\left((-\epsilon_0 c_3)\underline{a_z} - (\rho_s)\underline{a_z}\right) = 0$$

$$-\epsilon_0 c_3 - \rho_s = 0$$

$$c_3 = -\frac{\rho_s}{\epsilon_0}$$

$$-\frac{\rho_s}{\epsilon_0}a + c_4 = -\frac{\rho_s a}{\epsilon_1}$$

$$c_4 = \frac{\rho_s a}{\epsilon_0} - \frac{\rho_s a}{\epsilon_1}$$

$$\therefore \boxed{\Phi(z) = -\frac{\rho_s}{\epsilon_0}z + \frac{\rho_s a}{\epsilon_0} - \frac{\rho_s a}{\epsilon_1}} \qquad \boxed{E(z) = \frac{\rho_s}{\epsilon_0}\underline{a_z}} \text{ for } b > z > a$$

$$\Phi(z=b) = -\frac{\rho_s b}{\epsilon_0} + \rho_s\frac{a}{\epsilon_0} - \frac{\rho_s a}{\epsilon_1} = V_0$$

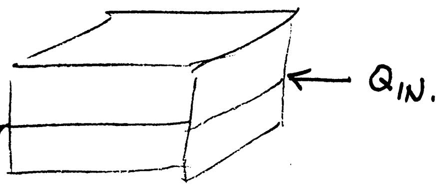
$$\rho_s\Big[\underbrace{\frac{a}{\epsilon_0} - \frac{b}{\epsilon_0}}_{\text{negative}} \underbrace{- \frac{a}{\epsilon_1}}_{\text{negative}}\Big] = V_0.$$

Thus $\rho_s < 0$

Note That ρ_s is actually < 0

3 (continued)

(b) Suppose one can inject free charge into the interface between the dielectrics (at $z=a$). How much charge must be injected to make the field in region 2 go to zero? Of what sign must the injected charge be?



to make this field zero make ϕ here equal to that of upper plate

$$\underline{n} \cdot (\underline{D}_2 - \underline{D}_1) = \rho_{IN} = \frac{Q_{IN}}{A}$$

$$\underline{a_z} \cdot (\phi - \rho_s \underline{a_z}) = \frac{Q_{IN}}{A}.$$

$$-\rho_s = \frac{Q_{IN}}{A}$$

$$Q_{IN} = -A\rho_s$$

(c) What is the capacitance?

$$C = \frac{Q}{V} = \frac{\rho_s A}{V} = \frac{\cancel{\rho_s} A}{\left[\frac{a}{\epsilon_0} - \frac{b}{\epsilon_0} - \frac{a}{\epsilon_1}\right]\cancel{\rho_s}.} = \frac{\epsilon_0 \epsilon_1 \quad A}{(a-b)\epsilon_1 - a\epsilon_0}$$

$$= \frac{\epsilon_0 \epsilon_1 A}{-b\epsilon_1 + a(\epsilon_1 - \epsilon_0)}$$

$$\frac{A}{\frac{a}{\epsilon_1} + \frac{1}{\epsilon_0}(b-a) + a\left(\frac{1}{\epsilon_1} - \frac{1}{\epsilon_0}\right)}$$

$$\frac{a}{\epsilon_1} + \frac{b}{\epsilon_0} - \frac{a}{\epsilon_0} + \frac{a}{\epsilon_1} - \frac{a}{\epsilon_0}$$

EEAP 210 ELECTROMAGNETIC FIELD THEORY

MARCH 28, 1983

EXAM NO. 3

NAME: ______________________________

Instructions:

1. Do ALL problems (there are five problems).
2. Write neatly
3. Read The problem carefully
4. THINK
5. Show all your work stating assumptions
6. Staple your formula sheet to The exam — it will be returned to you.

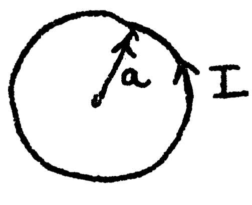
GRADING

PROBLEM	1		1. point possible
	2		1.5 pts possible
	3		2.0 pts. possible
	4		3.0 pts possible
	5		3.5 pts possible
TOTAL GRADE			11 pts. possible.

ADVISORY GRADE

1. If $B = \mu_0 H^2$ what is the induced magnetization m. (There is no permanent magnetization.) and the permeability μ? Is μ independent of H?

2. What is the vector potential at the center of a loop of wire carrying a current I? What is B?



3. A cyclindrical conductor of radius a oriented along the z-axis carries a non-uniformly distributed current

$$\underline{J} = J_0 r^2 \underline{a_z}$$

(a) What is the total current flowing in the conductor?

(b) What is H everywhere?

4. A infinitely long solenoid of radius a has n turns/meter and carries a current I.

(a) What is the surface current density?

(b) What is the on-axis z-directed field?

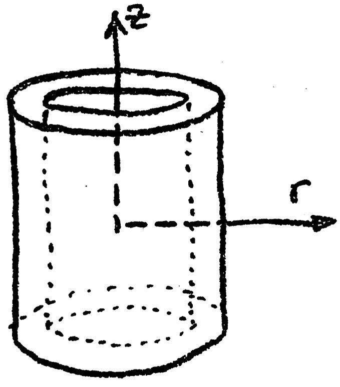
Hint: The on-axis z-directed field for a solenoid of length l is given by

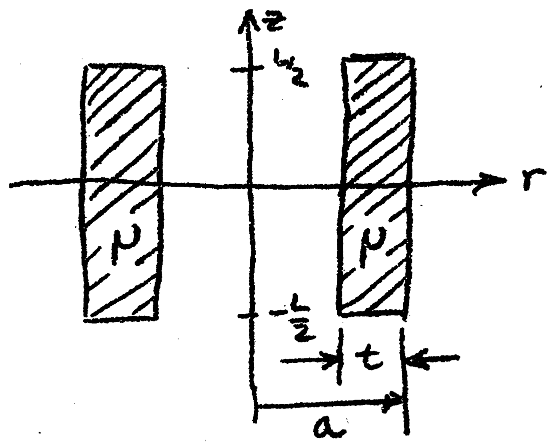
$$H_z = \frac{NI}{2\left(\frac{l}{4}+a^2\right)^{1/2}}$$

(c) What are the fields everywhere? State your assumptions.

HARD PROBLEM — DO LAST

5. Consider the hollow cyclinder constructed of a highly permeable material (such as iron) where $\mu >> \mu_0$ as shown below.





(a) This hollow cyclinder is placed in a uniform magnetic field $\underline{H} = H_0 \underline{a_z}$. What are $\underline{B}$ and $\underline{H}$ inside the permeable material. They may be assumed to be uniform.

(b) What is the Induced magnetization $\underline{M}$? Sketch it as a function of r.

(c) What are the equivalent surface currents $\underline{K}m$ for M? Sketch $\underline{Km}$ as a function of r.

(d) The z-component of the $\underline{B}$ field at the center of a cyclindrical current distribution $\underline{K} = K_0 \underline{a_\phi}$ of radius a and length L is given by:

$$B_z = \frac{\mu_0 K_0}{2}\left[\frac{-z+\frac{L}{2}}{\sqrt{(z-\frac{L}{2})^2+a^2}} + \frac{z+\frac{L}{2}}{\sqrt{(z+\frac{L}{2})^2+a^2}}\right]$$

Using this formula what is the field at $z=0$ on the axis of the cyclinder due to the currents found in (c)? Set up result — don't solve yet.

(e) If $a >> t$ reduce your results of (d).

(f) Does the induced field lie in the same direction as the applied field?

59 people took exam

EEAP 210 ELECTROMAGNETIC FIELD THEORY

MARCH 28, 1983

EXAM NO. 3

NAME: ______________________________

Instructions:

1. Do ALL problems (there are five problems).
2. Write neatly
3. Read the problem carefully
4. THINK
5. Show all your work stating assumptions
6. Staple your formula sheet to the exam — it will be returned to you.

GRADING

PROBLEM	1		1. point possible
	2		1.5 pts possible
	3		2.0 pts. possible
	4		3.0 pts possible
	5		3.5 pts possible
TOTAL GRADE			11 pts. possible.

ADVISORY GRADE ☐

1. If $B = \mu_o H^2$ what is the induced magnetization m. (There is no permanent magnetization.) and the permeability μ? Is μ independent of H?

by definition $B = \mu(H)\, H$

if $B = \mu_o H^2$ then $\mu_o H = \mu(H)$

furthermore, if $B = \mu_o (H + M) = \mu_o H^2$

$$\mu_o H + \mu_o m = \mu_o H^2$$

$\frac{1}{2}$ for each.

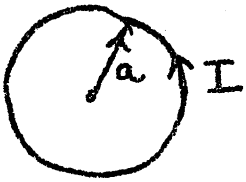
or $H + m = H^2$

$$m = H^2 - H$$

obviously, μ is a function of H.

2. What is the vector potential at the center of a loop of wire carrying a current I? What is B?

From our textbook $\underline{A} = \int_v \frac{\mu \underline{J}\, dv}{4\pi r}$



Recalling that $\underline{J} dv = \underline{I} dl$

$$\underline{A} = \int \frac{\mu \underline{I}\, dl}{4\pi r} = \frac{\mu}{4\pi} \int \frac{\underline{I}\, dl}{r}$$

where $\underline{I} = I\, \underline{a}_\phi$, $r = a$, $dl = a\, d\phi$

$$= \frac{\mu}{4\pi} \int_0^{2\pi} \frac{I \underline{a}_\phi\, a\, d\phi}{a} = \frac{\mu I}{4\pi} \int_0^{2\pi} \underline{a}_\phi\, d\phi$$

For this case $\int_0^{2\pi} \underline{a}_\phi\, d\phi = 0$. By inspection, $\underline{a}_\phi$ is a variable, not a constant and opposing current elements cancel each other,

Even though, $\underline{A} = 0$ (for $r = 0$) $\underline{B}$ is not necessarily zero. since by Biot Savart

$$H = \int \frac{I\, dl \times a_r}{4\pi r^2} = \oint_0^{2\pi} \frac{I\, a\, d\phi\, \underline{a}_\phi \times \underline{a}_r}{4\pi a^2}$$

$\frac{1}{2}$ point for integral.

$$= \frac{-I}{4\pi a} \underline{a}_z \int_0^{2\pi} d\phi = -\frac{I}{2a} \underline{a}_z$$

$$\underline{B = -\frac{\mu_o I}{2a}\, a_z}$$

3. A infinitely long round ~~cyclindrical~~ conductor of radius a oriented along the z-axis carries a non-uniformly distributed current

$$\underline{J} = J_o r^2 \underline{a_z}$$

(a) what is the total current flowing in the conductor?

(b) What is H everywhere?

$$\underline{I}_{total} = \int \underline{J}\, ds = \int_0^a \int_0^{2\pi} \underline{J}(r,\phi)\, r\,dr\,d\phi$$

$$= \int_0^a \int_0^{2\pi} \underline{a_z} J_o r^2\, r\,dr\,d\phi = \underline{a_z}\, J_o \int_0^a \int_0^{2\pi} r^3\,dr\,d\phi$$

$$= \underline{a_z}\, 2\pi J_o \int_0^a r^3\,dr = \underline{a_z}\, 2\pi J_o \frac{r^4}{4}\Big|_0^a$$

$$\underline{I}_{total} = \underline{a_z}\, \frac{\pi a^4}{2} J_o$$

use Ampère's Law $\oint \underline{H} \cdot \underline{dl} = \int \underline{J} \cdot \underline{ds}$

for $r > a$ $\quad H_\phi \cdot 2\pi r = I_o \quad$ where $I_o = \frac{\pi a^4}{2} J_o$

$$H_\phi = \frac{I_o}{2\pi r} = \frac{\pi a^4}{4\pi r} J_o = \frac{a^4}{4r} J_o$$

for $r \leq a$ $\quad H_\phi\, 2\pi r = 2\pi J_o \int_0^r r'^3\,dr' = 2\pi J_o \frac{r'^4}{4}\Big|_0^r$

$$H_\phi\, 2\pi r = \frac{\pi J_o}{2} r^4$$

$$H_\phi = \frac{J_o}{4} r^3$$

4. A infinitely long solenoid of radius a has n turns/meter and carries a current I.

(a) What is the surface current density?

(b) What is the on-axis z-directed field?

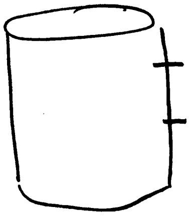
Hint: The on-axis z-directed field for a solenoid of length ℓ is given by

$$H_z = \frac{NI}{2\left(\frac{\ell^2}{4}+a^2\right)^{1/2}} \quad \text{error} \qquad N = \text{turns/meter}$$

(c) What are the fields everywhere? State your assumptions.

This problem was adapted from H.W.#5 **AND** the class lecture on it.

(a)



n turns/meter $\times$ I $\frac{\text{amps}}{\text{turn}}$ $=$ nI.

surface current density $\underline{K} = nI\,\underline{a}_\phi$

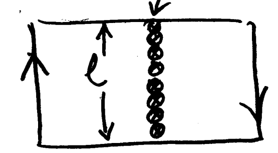
(b) $H_z \cong \frac{NI}{\ell}$ $\to$ nI.

total turns

length of coil

$\therefore$ $H_z = nI$

(c) Use Ampère's Law



do in two parts.

① Let H_z be at $r=0$.

Then $H_z(r=0)\ell - H_z(r>a)\ell = nI\ell$

where horizontal fields are neglected.

$\therefore$ $H_z(r=0) - H_z(r>a) = nI$.

$nI - H_z(r>a) = nI \Rightarrow H_z(r>a) = 0$

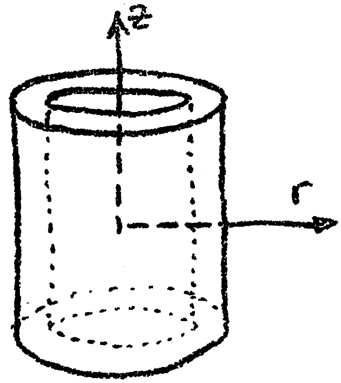
② If $H_z(r>a)=0$. Then pick any $r<a$

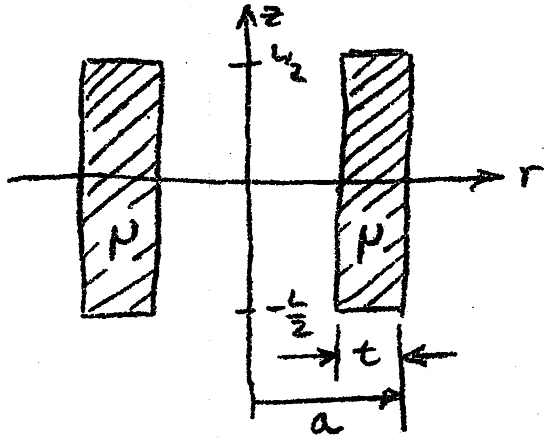
by Ampère's Law $H_z(r<a) = nI$ $\therefore$ H_z is constant inside the coil.

$$\therefore \quad H_z = \begin{cases} nI & r \le a \\ 0 & r>a \end{cases} \qquad B_z = \begin{cases} \mu_0 nI & r \le a \\ 0 & r>a \end{cases}$$

HARD PROBLEM — DO LAST

5. Consider the hollow cyclinder constructed of a highly permeable material (such as iron) where $\mu \gg \mu_0$ as shown below.





(a) This hollow cyclinder is placed in a uniform magnetic field $\underline{H} = H_0 \underline{a_z}$. What are $\underline{B}$ and $\underline{H}$ inside the permeable material. They may be assumed to be uniform.

(b) What is the induced magnetization $\underline{M}$? Sketch it as a function of r.

(c) What are the equivalent surface currents $\underline{K}_m$ for M? Sketch $\underline{K_m}$ as a function of r.

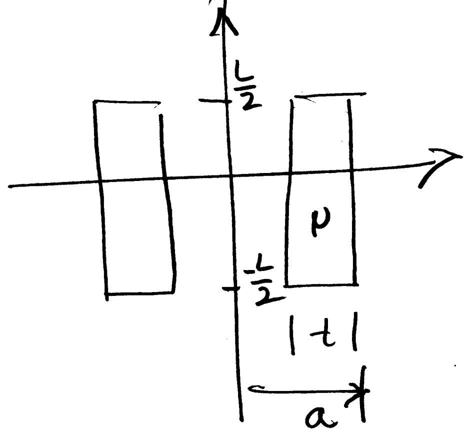
(d) The z-component of the $\underline{B}$ field at the center of a cyclindrical current distribution $\underline{K} = K_0 \underline{a_\phi}$ of radius a and length L is given by:

$$B_z = \frac{\mu_0 K_0}{2}\left[\frac{-z+\frac{L}{2}}{\sqrt{(z-\frac{L}{2})^2+a^2}} + \frac{z+\frac{L}{2}}{\sqrt{(z+\frac{L}{2})^2+a^2}}\right]$$

Using this formula what is the field at $z=0$ on the axis of the cyclinder due to the currents found in (c)? Set up result — don't solve yet.

(e) If $a \gg t$ reduce your results of (d).

(f) Does the induced field lie in the same direction as the applied field?



(a) applied field everywhere in free space is $H_0 \underline{a_z}$

If I drop this structure in place at the origin of a cyclindrical coordinate system, the applied H field will remain the same everywhere except possibly in the permeable material.

For the permeable material, normal B is continuous.

$\therefore$ $\mu_0 H_0 \underline{a_z} = \mu H \underline{a_z}$ where H is the field in the material.

$H = \frac{\mu_0 H_0}{\mu}$.

This is uniform throughout the permeable material.

Likewise, $\underline{B} = \mu_0 H_0 \underline{a_z}$ outside, and $B = \mu H^{a_z} = \mu_0 H_0 a_z$ inside the material.

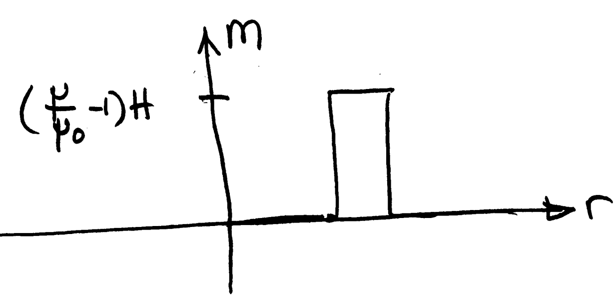
$B = \mu_0 H_0 \underline{a_z}$, $H = \frac{\mu_0}{\mu} H_0 \underline{a_z}$ inside the permeable material.

(b) by definition $B = \mu_0 (H + M)$ in the permeable material.

letting $\mu_0 (H + M) = \mu H$

$$\mu_0 H + \mu_0 M = \mu H$$

$$M = \frac{\mu H - \mu_0 H_0}{\mu_0} = \left(\frac{\mu}{\mu_0} - 1\right) H$$

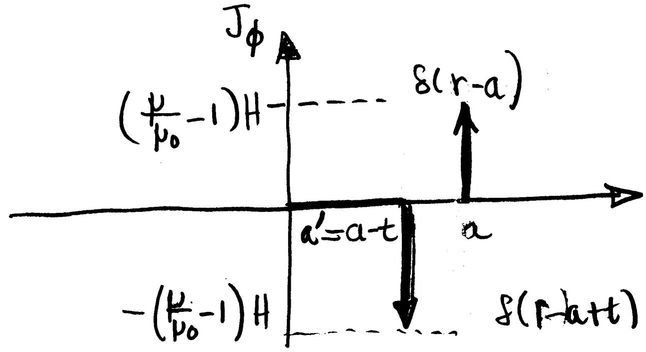


(c) The easiest way to find $\underline{J_m}$ is to recall $\underline{J_m} = \underline{\nabla} \times \underline{M}$

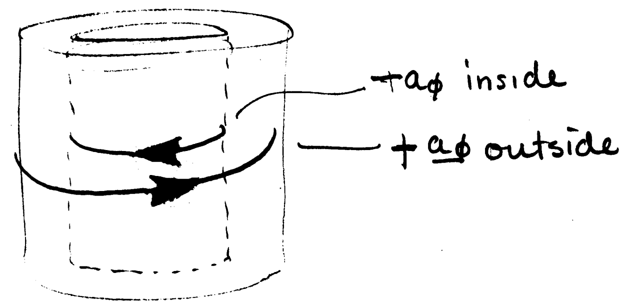
This is nothing more than the derivative of $\underline{M}$

as. $$\underline{\nabla} \times \underline{M} = \underline{a}_r \frac{1}{r} \frac{\partial M_z}{\partial \phi} - \underline{a}_\phi \frac{\partial M_z}{\partial r}$$

Well, $M_z = 0$ and $\underline{\nabla} \times \underline{M} = -\underline{a}_\phi \frac{dM_z}{dr}$



This looks like two coaxial current distributions.



(d) $\Delta B_z = B_z(J_\phi(a-t)) + B_z(J_\phi(a))$

for $z=0$ $$B_z = \frac{\mu_0 K_0}{2}\left[\frac{\frac{L}{2}}{\sqrt{\frac{L^2}{4}+a^2}} - \frac{-\frac{L}{2}}{\sqrt{\frac{L^2}{4}+a^2}}\right]$$

$$= \mu_0 K_0 L \frac{1}{\sqrt{\frac{L^2}{4}+a^2}}$$

and substitute in for each a as shown above.

$$\Delta B_z = \mu_0 \left(\frac{\mu}{\mu_0}-1\right) H L \left[\frac{-1}{\sqrt{\frac{L^2}{4}+(a-t)^2}} + \frac{+1}{\sqrt{\frac{L^2}{4}+a^2}}\right].$$

(e) consider just the factor in square brackets in (d)

$$\frac{-1}{\sqrt{\frac{L^2}{4}+(a-t)^2}} + \frac{+1}{\sqrt{\frac{L^2}{4}+a^2}}$$

the correct way to do this is to note that we want to factor out a common expression

$$\frac{-1}{\sqrt{\frac{L^2}{4}+a^2-2at+t^2}} + \frac{+1}{\sqrt{\frac{L^2}{4}+a^2}} = \frac{-1}{\sqrt{\frac{L^2}{4}+a^2}\left(1+\frac{t^2-2at}{\frac{L^2}{4}+a^2}\right)^{1/2}} + \frac{+1}{\sqrt{\frac{L^2}{4}+a^2}}$$

$$= \frac{1}{\sqrt{\frac{L^2}{4}+a^2}}\left[\frac{-1}{1+\frac{1}{2}\frac{t^2-2at}{\frac{L^2}{4}+a^2}} + 1\right]$$

as $t \ll a$ $\quad t^2-2at \approx 2at$

$$\cong \frac{1}{\sqrt{\frac{L^2}{4}+a^2}}\left[\frac{-1}{1-\frac{1}{2}\frac{2at}{\frac{L^2}{4}+a^2}}+1\right] = \frac{1}{\sqrt{\frac{L^2}{4}+a^2}}\left[-1-\frac{1}{2}\frac{2at}{\left(\frac{L^2}{4}+a^2\right)}+1\right]$$

$$= \frac{-at}{\left(\frac{L^2}{4}+a^2\right)}$$

$$\therefore \Delta B_z = -\frac{\mu_0\left(\frac{\mu}{\mu_0}-1\right)HL\,at}{\left(\frac{L^2}{4}+a^2\right)^{3/2}}$$

(f) No, in the opposite direction. This is the principle of magnetic shielding. As t becomes larger, ΔB_z becomes more negative acting to cancel out the applied H field and make the field inside a shielded volume zero. Note that we only looked at the field <u>ON</u> the z-axis, at $z=0$. Otherwise, our expressions would have become <u>EXTREMELY COMPLICATED</u>.

EEAP 210

Exam #4

APRIL 15, 1983

Name ______________________________

PROBLEM		
1		3
2		3
3		2
4		3
TOTAL		11

ADVISORY GRADE

Exam No. 4
SOLUTION SET

1. An electromagnetic wave propagates in a medium described by the propagation constant $\gamma^2 \triangleq j\omega\mu(\sigma + j\omega\epsilon)$.

(a) Define and/or discuss displacement and conduction current as they relate to the above expression for γ^2.

(b) For $\omega\epsilon \gg \sigma$ and $\omega\epsilon \ll \sigma$ what are approximate forms for $\gamma = \alpha + j\beta$.

(c) If $\sigma = 5.7 \times 10^7 \, (\Omega\text{-m})^{-1}$, $\omega = 2\pi \times 10^8/\text{sec}$, and $\epsilon = \frac{1}{36\pi} \times 10^{-9}$ farad/meter, is the medium a conductor or an insulator.

(d) Calculate α and β for the values given in (c).

(e) What is $\hat{\eta}$, the impedance of the medium, for the values given in (c).

(f) If $\underline{\hat{E}} = E^- e^{\alpha z + j\beta z} \, a_y$ corresponds to a plane wave propagating in the $-z$ direction, what is $\underline{\hat{H}}$?

(a) displacement current $= \dfrac{\partial D}{\partial t}$

conduction current $= \sigma E$

for complex fields $\dfrac{\partial \hat{D}}{\partial t} \rightarrow j\omega\epsilon \hat{E}$

and we see that the first term σ corresponds to conduction current, The second term $j\omega\epsilon$ corresponds to displacement current

(b) it helps to rewrite γ^2 to answer this question

$$\gamma^2 = j\omega\mu\sigma - \omega^2\mu\epsilon = -\omega^2\mu\epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)$$

if $\omega\epsilon \gg \sigma$ the second term is much less than 1 and

$$\gamma^2 \approx -\omega^2\mu\epsilon$$

or $\gamma \approx j\omega\sqrt{\mu\epsilon}$

for this case $\alpha \approx 0$ $\beta \approx \omega\sqrt{\mu\epsilon}$

if $\omega\epsilon \ll \sigma$ Then $\frac{\sigma}{\omega\epsilon} \gg 1$ and the second term dominates

$$\gamma^2 \approx -\omega^2\mu\epsilon\left(-j\frac{\sigma}{\omega\epsilon}\right) = j\omega\mu\sigma$$

$$\gamma = \sqrt{j}\sqrt{\omega\mu\sigma}$$

Recall that $j = e^{j\frac{\pi}{2}}$ so $\sqrt{j} = e^{j\frac{\pi}{4}} = \cos 45^\circ + j\sin 45^\circ$

$$= \frac{1+j}{\sqrt{2}}.$$

(c) $\omega\epsilon = (2\pi\times10^8)\left(\frac{1}{36\pi}\times10^{-9}\right) = 5.56\times10^{-3}$

$\sigma = 5.7\times10^7$

$\therefore$ $\sigma \gg \omega\epsilon$ and the material is a conductor because conduction current dominates

(d) use results of (b)

$$\gamma = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}} = (1+j)\sqrt{\frac{(2\pi\times10^8)(4\pi\times10^{-7})(5.7\times10^7)}{2}}$$

$$= (1+j)(1.487\times10^5)$$

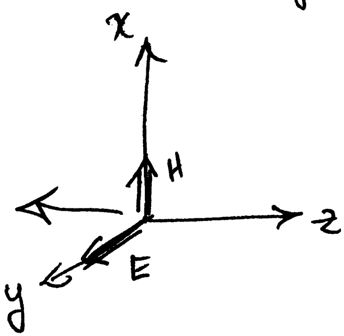
$\therefore$ $\alpha = 1.487\times10^5$

$\beta = 1.487\times10^5$

(e) $\hat{\eta} = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j}\sqrt{\omega\mu\sigma}} = \sqrt{j}\sqrt{\frac{\omega\mu}{\sigma}} = \frac{1+j}{\sqrt{2}}\sqrt{\frac{(2\pi\times10^8)(4\pi\times10^{-7})}{5.7\times10^7}}$

$= (1+j)(2.655\times10^{-3})$

(f) for the coordinate system shown below.



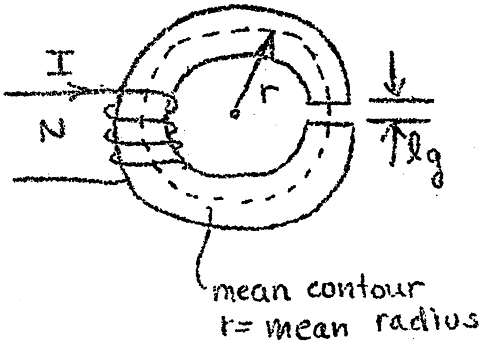
the H field for travel in the $-z$ direction must be.

$$\underline{\hat{H}} = -\frac{\hat{E}}{\hat{\eta}}\,\underline{a}$$

$$= -\frac{E^{-} e^{\alpha z + j\beta z}}{(1+j)(3.7218 \times 10^{-3})}\,\underline{a}$$

$$= (1-j)\,134.3\,E^{-}\,e^{\alpha z + j\beta z}$$

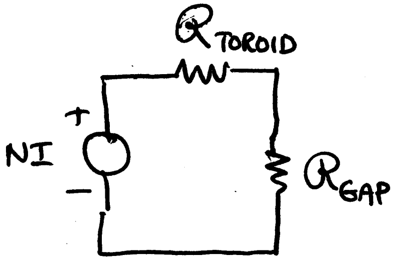
2. It is desired to produce a magnetic field $B = 0.1\ \text{Wb/m}^2$ in the air gap of the electromagnetic shown below. The cross-section of the toroid is $1\ \text{cm}^2$, its <u>mean</u> radius is 10 cm, and the gap length is 0.1 cm. Assume the permeability of the ring is $\mu = 4000\,\mu_0$ and that fringing can be ignored.



(a) Draw the equivalent magnetic circuit.

(b) Find the number of turns, N, required to produce this field.

Equivalent circuit



$A = 1\ \text{cm}^2 = 10^{-4}\ \text{m}.$

$$\mathcal{R}_{GAP} = \frac{l_g}{\mu_0 A} = \frac{0.1\times10^{-2}\ \text{m}}{(4\pi\times10^{-7})(10^{-4})} = \frac{0.1\times10^{-2}}{12.6\times10^{-11}} = 7.96\times10^{6}\ \frac{\text{A-t}}{\text{Wb}}.$$

$$\mathcal{R}_{TOROID} = \frac{l_{TOROID}}{\mu A} = \frac{2\pi(0.1)}{4000\,\mu_0 A} = \frac{0.628}{4000\,(12.6\times10^{-11})} = \frac{0.628}{5.04\times10^{-7}}$$

$$= 0.125\times10^{7} = 1.25\times10^{6}\ \frac{\text{A-t}}{\text{Wb}}$$

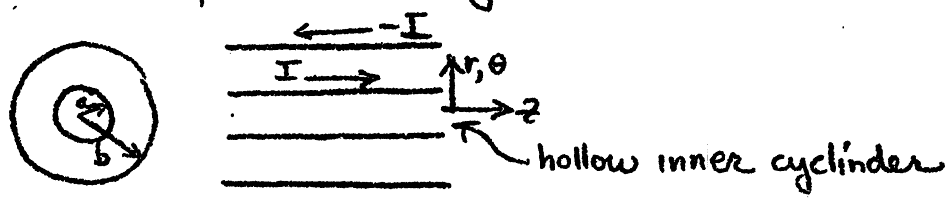
$$NI = \phi\left[\mathcal{R}_{GAP} + \mathcal{R}_{TOROID}\right] \quad \text{where } \phi = \int B\cdot n\,da = BA$$

$$= (0.1)(10^{-4}) = 10^{-5}$$

$$NI = (10^{-5})\left(7.96\times10^{6} + 1.25\times10^{6}\right].$$

$$= 9.21\times10^{1} = 92.1 \text{ Ampere turns.}$$

3. A coaxial transmission line is made up of two thin-walled conducting cyclinders with radii a and b. A current I flows along the inner cyclinder and a return current −I along the outer cyclinder. Calculate the inductance per unit length.



to calculate the inductance per unit length we need H and B.

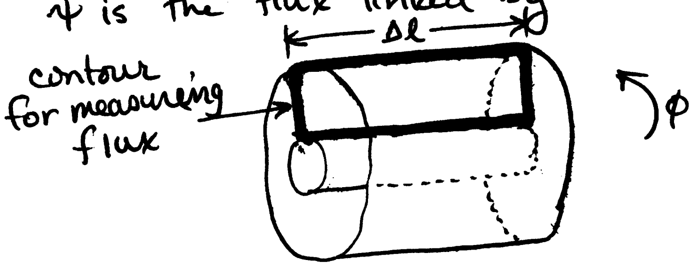
From Ampère's Law $\oint H \cdot dl = \int J \cdot ds = I$

$$H \cdot 2\pi r = I$$

$$H_\phi = \frac{I}{2\pi r}$$

$$B_\phi = \frac{\mu_0 I}{2\pi r}$$

the inductance L is defined to be $L = \frac{\psi}{I}$ where ψ is the flux linked by the contour.



$$\psi = \int B(r)\, dr\, \Delta l = \Delta l \frac{\mu_0 I}{2\pi} \int_a^b \frac{dr}{r}$$

$$\psi = \frac{\mu_0 I}{2\pi} \Delta l \ln\left(\frac{b}{a}\right)$$

$$L = \frac{\psi}{I} = \frac{\mu_0}{2\pi} \Delta l \ln\left(\frac{b}{a}\right)$$

$$\frac{L}{\Delta l} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right)$$

Alternatively,

since $$H_\phi = \begin{cases} \dfrac{I}{2\pi r} & a<r<b \\ 0 & \text{elsewhere.} \end{cases}$$ (no fields inside conductor)

$$W_m = \frac{1}{2}\mu \int |\underline{H}|^2 dv$$

$$= \frac{1}{2}\mu \int_0^{2\pi}\int_a^b \left(\frac{I}{2\pi r}\right)^2 r\,dr\,d\phi$$

$$= \frac{1}{2}\mu \; 2\pi \int_a^b \frac{I^2}{4\pi^2 r^2} r\,dr$$

$$= \frac{\pi\mu I^2}{4\pi^2}\int_a^b \frac{dr}{r}$$

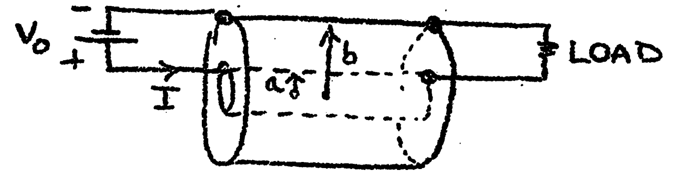
$$= \frac{\mu I^2}{4\pi} \ln b/a$$

but $W_m = \frac{1}{2} L I^2$

$$\therefore \quad \frac{1}{2} L I^2 = \frac{\mu I^2}{4\pi} \ln b/a$$

or $$L = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)$$

4. The power transmitted by a lossless coaxial cable can be considered in terms of the Poynting vector inside the dielectric medium between the inner and outer conductors. Assume that a voltage V_o is applied between the inner conductor (of radius a) and the outer conductor (of radius b) causes a current I to flow to a load impedance.



(a) Find the fields in the dielectric. Hint: Recall $\phi = -\int \underline{E} \cdot d\underline{l}$

(b) In what direction is power transmitted?

(c) What is the total power transmitted?

(d) Consider the situation where the dielectric becomes lossy and a current flows radially between the inner and outer conductors. <u>Discuss</u> if this would alter the Poynting vector. How?

(a) From problem (3) $H\phi = \dfrac{-I}{2\pi R}$ where $I = \dfrac{V_o}{R_{LOAD}}$ (no transmission line effects)

For E recall $\nabla^2 \Phi = 0$ and $E = -\nabla\Phi = -\dfrac{d\Phi}{dr}$

for cylindrical coordinates

$$\nabla^2 \Phi = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\cancelto{0}{\frac{\partial^2 \Phi}{\partial \phi^2}} + \cancelto{0}{\frac{\partial^2 \Phi}{\partial z^2}} = 0$$

because of symmetry

because uniform in z-direction

$$\therefore \frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = 0$$

$$r\frac{d\Phi}{dr} = c_1 \quad \text{but} \quad \frac{d\Phi}{dr} = -E$$

and $\dfrac{d\Phi}{dr} = \dfrac{c_1}{r} \Rightarrow \underline{E} = -\dfrac{c_1}{r}\,\underline{a_r}$

to get the constant

$$\Phi = -\int_a^b \underline{E} \cdot \underline{a_r}\, dr$$

$$\Phi(b) - \Phi(0) = -\int_a^b -\frac{c_1}{r}\, dr = \int_a^b c_1 \frac{dr}{r} = c_1 \ln(b/a)$$

$$\therefore \quad c_1 = \frac{\Phi(b) - \Phi(a)}{\ln(b/a)} = \frac{-V_0}{\ln(b/a)}$$

and $$\underline{E} = \frac{V_0}{r \ln(b/a)} \underline{a_r}$$

$$\underline{H} = \frac{-V_0}{R_{LOAD}\, 2\pi r} \underline{a_\phi}$$

(b) $$\underline{S} = \underline{E} \times \underline{H} = \frac{V_0^2}{R_{LOAD}} \frac{1}{2\pi r^2 \ln(b/a)} \underline{a_r} \times (-\underline{a_\phi})$$

$$= \frac{V_0^2}{R_{LOAD}} \frac{1}{2\pi r \ln(b/a)} \underline{a_z}$$

(c) $$\text{power} = \int \underline{S} \cdot \underline{ds} = \frac{V_0^2}{R_{LOAD}} \int_a^b \int_0^{2\pi} \frac{1}{2\pi r^2 \ln(b/a)}\, r\, dr\, d\phi$$

$$= \frac{V_0^2}{R_{LOAD}} \int_a^b \int_0^{2\pi} \frac{d\phi\, dr}{2\pi r \ln b/a}$$

$$= \frac{V_0^2}{R_{LOAD}} \frac{2\pi[\ln b - \ln a]}{2\pi r \ln b/a} = \frac{V_0^2}{R_{LOAD}}$$

THE D.C. RESULT...

(d) If medium becomes lossy power flow <u>also</u> in direction of radial current flow. So $\underline{S}$ has radial and longitudinal components.

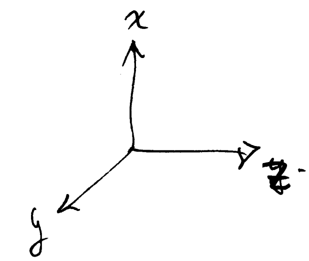
EEAP 210

EXAM NO. 4 (MAKE-UP)

SOLUTION SET

PROBLEM	SCORE	POINTS
PROPAGATION		3
MAGNETIC CIRCUITS		3
INDUCTANCE		3
FARADAY'S LAW		2
TOTAL		11

ADVISORY GRADE



PROPAGATION

An electric field is of the form

$$\underline{E} = 100\, e^{j(2\pi\times10^{6}t + 2\pi\times10^{-2}z)}\ \underline{a_x} \quad \frac{\text{volts}}{m.}$$

(a) What is the frequency and wavelength of the wave?

(b) If the medium is a lossless dielectric with ϵ_0 error $= 4\pi\times10^{-7}$ H/m and $\mu = 4\mu_0$ ($\mu_0 = \frac{1}{36\pi}\times10^{-9}$ F/m), what is the wave impedance η and the magnetic field H?

(c) What is the time-averaged power per unit area carried by the wave?

(d) If the dielectric becomes lossy with $\sigma = 1$/ohm-m, write the expression for γ^2. Identify conduction and dielectric terms. Find α and β.

(a)
$$\omega = 2\pi\times10^{6} \qquad \hat{\underline{E}}^{+} = \hat{E}_0^{+} e^{j(\omega t - \beta z)}\ \underline{a_x}$$
$$= 2\pi f$$
$$\therefore f = 10^{6}$$
$$\beta = \frac{2\pi}{\lambda} = 2\pi\times10^{-2} \quad \therefore \lambda = 10^{2} \text{ meters}$$

(b)
$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\frac{4}{36\pi}\times10^{-9}\ F/m}{4\pi\times10^{-7}\ H/m}} = \sqrt{\frac{1}{36\pi^2}\times10^{-2}}\ \Omega$$
$$= .0530\times10^{-1} = 5.3\times10^{-3}\,\Omega$$

(c)
$$H = -\frac{\hat{E}_0^{+}}{\eta} e^{j(\omega t - \beta z)}\ \underline{a_y}$$

(c)
$$S = \tfrac{1}{2}\mathrm{Re}(E\times H^*) = \tfrac{1}{2}\mathrm{Re}\left[\hat{E}_0^{+}\cancel{e^{-j\beta z}} \cdot -\frac{\hat{E}_0^{+*}}{\eta}\cancel{e^{+j\beta z}}\ \underline{a_z}\right]$$
$$= \frac{1}{2}\frac{|E_0|^2}{\eta}a_z = \frac{1}{2}\frac{(100)^2}{5.3\times10^{-3}}a_z = \frac{10^{7}}{10.6}\ \text{watts}/m^2$$
$$= .943\times10^{6}\ \text{watts}/m^2 = 943\ \text{kw}/m^2$$

(d) $\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$

$= j\omega\mu\, j\omega\epsilon\left(\frac{\sigma}{j\omega\epsilon} + 1\right)$

$= -\omega^2\mu\epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)$

what is $\frac{\sigma}{\omega\epsilon} = \frac{1}{(2\pi\times10^6)(4\pi\times10^{-7})} = \frac{1}{8\pi^2}\times10^1$

$= .01267\times10 = 0.1267$

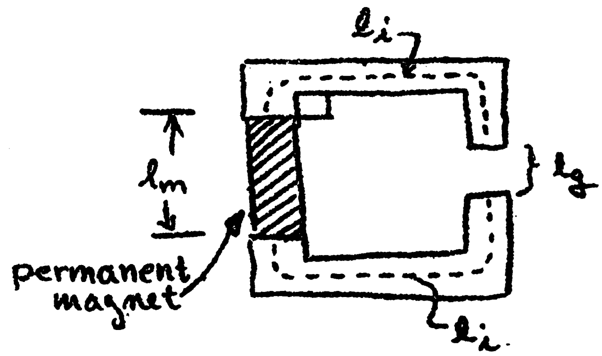
$\omega^2\mu\epsilon = (2\pi\times10^6)^2\left(\frac{\cancel{1}}{\cancel{36\pi}_{9}}\times10^{-9}\right)(4\cancel{\pi}\times10^{-7})$

$= \frac{8\pi}{9}\times10^{12-9-7} = \frac{8\pi}{9}\times10^{-4} = 2.79\times10^{-4}$

$\therefore\ \gamma^2 = -2.79\times10^{-4}(1 - j\,0.1267)$

this is not expandable $\frac{\sigma}{\omega\epsilon}$ is <u>NOT</u> $\gg$ or $\ll 1$.

MAGNETIC CIRCUITS



iron arms $\mu = 4000\mu_0$

A permanent magnet is used in the magnetic circuit shown above. The cross sectional area is A, and is uniform for the magnet, iron arms, and gap.

(a) Draw an equivalent electrical circuit for the magnetic circuit.

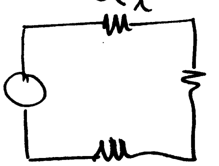
(b) Assume the magnet produces a magnetic field intensity H. What is the magnetic field intensity across the gap?

(c) What is the flux in the gap?

(d) If the magnetic potential ($\mathcal{F}$) across the iron arms can be neglected, what is the relationship between the magnetic field intensity and the flux in the gap.

(e)

(a) $Hl_m = \mathcal{F}$



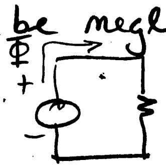
(b) $$Hl_m + 2Hl_i + H_g l_g = 0. \qquad \text{since} \quad \int H \cdot dl = NI$$

$$H_g = -\frac{Hl_m + 2Hl_i}{l_g} = -H\,\frac{l_m + 2l_i}{l_g}.$$

(c) $$\Phi_g = BA = \mu_0 H_g A = -\mu_0 H\left(\frac{l_m + 2l_i}{l_g}\right)A$$

(d) if R_i can be neglected, we have.

Hl_m



and $\mathcal{F} = \Phi_g R_g$.

$$Hl_m = -B\left[\frac{l_m + 2l_i}{\cancel{l_g}}\right]\cancel{A}\left[\frac{\cancel{l_g}}{\mu_0 \cancel{A}}\right]$$

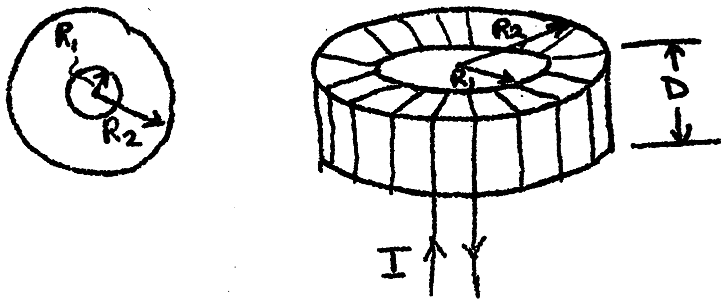
or directly from above.

$$B = -\mu_0 H\,\frac{l_m + 2l_i}{l_g} \qquad ?$$

$$\therefore Hl_m = -B\left[\frac{l_m + 2l_i}{\mu_0}\right]$$

INDUCTANCE

What is the inductance of a toroidal coil consisting of N turns on a core of permeability μ. The core has a rectangular cross-section.



$$\oint H \cdot dl = H_\phi \, 2\pi r = \begin{cases} 0 & r < R_1 \\ Ni & R_1 < r < R_2 \\ 0 & r > R_2 \end{cases} \qquad H_\phi = \frac{Ni}{2\pi r}$$

$$\text{single loop flux} = \Phi = \mu \int H \cdot ds$$

$$= \mu \int \frac{Ni}{2\pi} \frac{ds}{r}$$

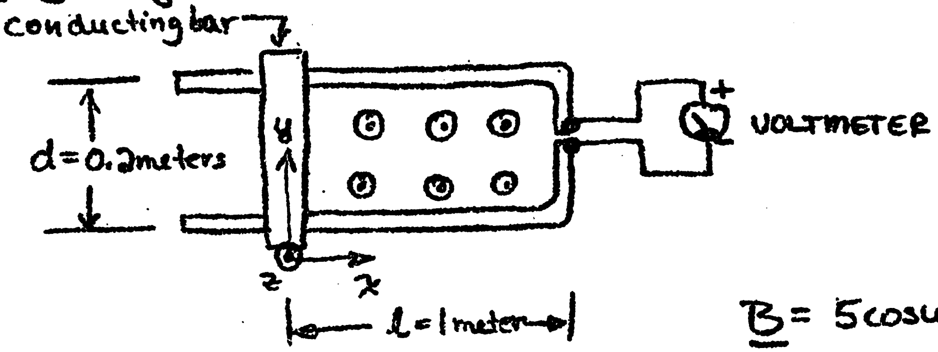
$$= \mu \int_{R_1}^{R_2} \frac{Ni}{2\pi} \frac{D\,dr}{r} = \frac{\mu N i D}{2\pi} \int_{R_1}^{R_2} \frac{dr}{r}$$

$$= \frac{\mu N i D}{2\pi} \left[\ln R_2 - \ln R_1 \right]$$

$$L = \frac{N\Phi}{i} = \frac{\mu D N^2 \cancel{i}}{2\pi \cancel{i}} \ln\left(R_2/R_1\right) = \frac{\mu N^2 D \ln\left(R_2/R_1\right)}{2\pi}$$

FARADAY'S LAW

A conducting, sliding bar oscillates (travels) back and forth ~~between~~ on two parallel conducting rails in a sinusoidally varying magnetic field.



$\underline{B} = 5\cos\omega t \; \underline{a_z}$

The position of the sliding bar is given by $x = 0.35\cos\omega t$
What voltage does the voltmeter read?

$$V_o = -\frac{\partial \Phi}{\partial t} = -\frac{\partial}{\partial t}\int B \cdot ds$$

$$= -\frac{\partial}{\partial t}\left[\underbrace{B}_{\text{field}} \cdot \Big(a_z \underbrace{d\,(1 + 0.35\cos\omega t)}_{\text{area}}\Big)\right]$$

$$= -\frac{\partial}{\partial t}\; 5\cancel{\cos\omega t}\; \cancel{a_z} \cdot \cancel{a_z}\; 0.2\cancel{\text{meters}}\,(1 + 0.35\cos\omega t)$$

$$= -\frac{\partial}{\partial t}\left[\cos\omega t\,(1 + 0.35\cos\omega t)\right]$$

$$= -\frac{\partial}{\partial t}\left(\cos\omega t + 0.35\cos^2\omega t\right)$$

$$= -\left[-\omega\sin\omega t - 0.7\omega\cos\omega t\sin\omega t\right]$$

$$= \omega\sin\omega t\,[1 + 0.7\cos\omega t].$$

60 people took exam.

EEAP 210 ELECTROMAGNETIC FIELD THEORY

APRIL 29, 1983

EXAM NO. 5

NAME: SOLUTIONS

Instructions:

1. Do **ALL** problems (there are three problems)
2. Write neatly.
3. Show all your work — indicate where something you used came from.

$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$

$e^{j\theta} + e^{-j\theta} = 2\cos\theta$

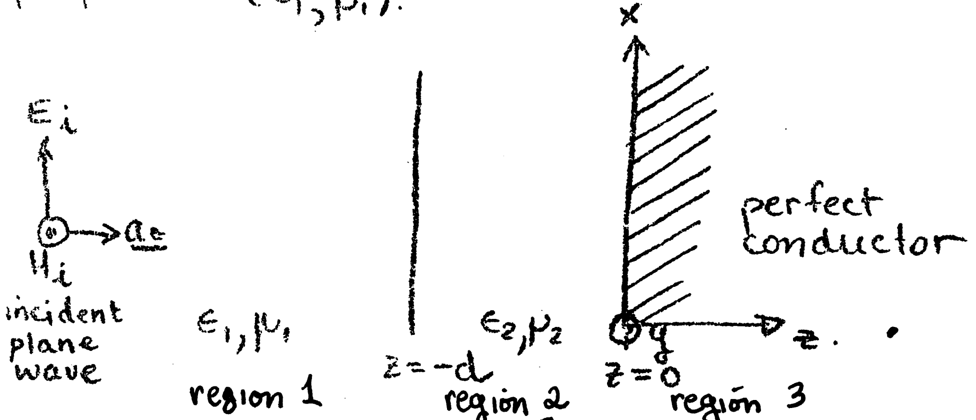
GRADING

PROBLEM 1		4
2		4
3		3
TOTAL		11

ADVISORY GRADE

1. REFLECTION & TRANSMISSION

A dielectric (ϵ_2, μ_2) of thickness d coats a perfect conductor. A uniform plane wave is normally incident onto the coating from the surrounding medium with properties (ϵ_1, μ_1).



(a) write expressions for all the waves in all the media. Identify any transmitted or reflected waves as such.

(b) what are the boundary conditions at $z=0$, $z=-d$?

(c) What is the reflection coefficient at $z=0$, $z=-d$?

(d) Set up expressions for the total fields for $z<-d$, $-d<z<0$, $z>0$. Apply the boundary conditions of (b) to set up equations for solving for the reflection and transmission coefficients. DO NOT SOLVE THESE EQUATIONS

(a)

	region 1	region 2	region 3
E	$E_i e^{-j\beta_1 z} + E_r e^{+j\beta_1 z}$	$\hat{E}^+ e^{-j\beta_2 z} + \hat{E}^- e^{+j\beta_2 z}$	0
H	$\underbrace{\dfrac{E_i e^{-j\beta_1 z}}{\eta_1}}_{\text{incident}} - \underbrace{\dfrac{E_r e^{j\beta_1 z}}{\eta_1}}_{\text{reflected from dielectric}}$	$\underbrace{\dfrac{\hat{E}^+ e^{-j\beta_2 z}}{\eta_2}}_{\text{transmitted through dielectric}} - \underbrace{\dfrac{\hat{E}^- e^{+j\beta_2 z}}{\eta_2}}_{\text{reflected from metal.}}$	0

(b)

$\left.\begin{array}{l}\text{tangential } E \text{ continuous} \\ \text{tangential } H \text{ continuous}\end{array}\right\}$ at $z=-d$

$\left.\begin{array}{l}\text{tangential } E = 0 \quad \text{(no charge on surface)} \\ \text{tangential } H = K \quad \text{(surface current)}\end{array}\right\}$ $z=0$.

(c) $\Gamma_{21} = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

for a lossless dielectric $\eta_1 = \sqrt{\dfrac{\mu_1}{\epsilon_1}}$, $\eta_2 = \sqrt{\dfrac{\mu_2}{\epsilon_2}}$

for a perfect conductor $\sigma \to \infty$ and $\eta_3 \to 0$.

$\therefore$ at $z = -d$ $\quad \Gamma_d = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ where $\eta_1 = \sqrt{\dfrac{\mu_1}{\epsilon_1}}$, $\eta_2 = \sqrt{\dfrac{\mu_2}{\epsilon_2}}$

$z = 0 \qquad \Gamma = -\dfrac{\eta_2}{\eta_2} = -1$

(d) do conductor $(-d < z < 0)$ first

$$\begin{aligned} E_2 &= \hat{E}^+ e^{-j\beta_2 z} + \hat{E}^- e^{+j\beta_2 z} \\ &= \hat{E}^+ (e^{-j\beta_2 z} + \Gamma e^{+j\beta_2 z}) \\ &= \hat{E}^+ (e^{-j\beta_2 z} - e^{+j\beta_2 z}) = \hat{E}^+ (-2j \sin\beta_2 z) \end{aligned}$$

Hint: solve for fields in region 2 first.

$$\begin{aligned} H_2 &= \hat{H}^+ e^{-j\beta_2 z} + \hat{H}^- e^{+j\beta_2 z} \\ &= \frac{\hat{E}^+ e^{-j\beta_2 z}}{\eta_2} - \frac{\hat{E}^- e^{j\beta_2 z}}{\eta_2} = \frac{\hat{E}^+}{\eta_2}\left[e^{-j\beta_2 z} - e^{-j\beta_2 z}\right] \\ &= \frac{\hat{E}^+}{\eta_2} 2\cos\beta_2 z. \end{aligned}$$

then: ① $\hat{E}_i(e^{-j\beta_1 d} + \Gamma_d e^{j\beta_1 d}) = \hat{E}^+ 2j \sin\beta_2 d$ ② $\dfrac{\hat{E}_i(e^{-j\beta_1 d} - \Gamma_d e^{j\beta_1 d})}{\eta_1} = \dfrac{\hat{E}^+ 2\cos\beta_2 d}{\eta_2}$

① $\hat{E}_i e^{-j\beta_1 d} + \Gamma_d \hat{E}_i e^{j\beta_1 d} = \hat{E}^+ 2j \sin\beta_2 d$

② $E_i e^{-j\beta_1 d} - \Gamma_d \hat{E}_i e^{j\beta_1 d} = \dfrac{\eta_1}{\eta_2} 2\hat{E}^+ \cos\beta_2 d.$ multiply ② by -1 and subtract.

$$2\Gamma_d \hat{E}_i e^{j\beta_1 d} = 2\hat{E}^+ \left(j \sin\beta_2 d - \frac{\eta_1}{\eta_2}\cos\beta_2 d\right)$$

$$\frac{E^+}{E_i} = \frac{\Gamma_d e^{j\beta_1 d}}{j\sin\beta_2 d - \frac{\eta_1}{\eta_2}\cos\beta_2 d}.$$

multiplying by $+1$ and subtracting

$$2\hat{E}_i e^{-j\beta_1 d} = 2\hat{E}^+ \left(j\sin\beta_2 d + \frac{\eta_1}{\eta_2}\cos\beta_2 d\right)$$

$$\frac{\hat{E}^+}{E_i} = \frac{e^{-j\beta_1 d}}{j\sin\beta_2 d + \frac{\eta_1}{\eta_2}\cos\beta_2 d}.$$

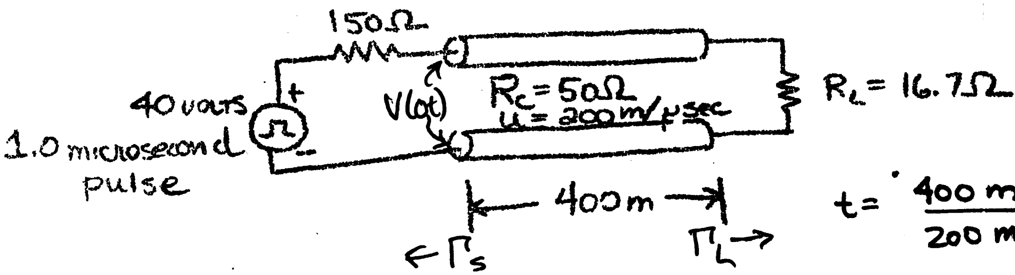
2. PULSED TRANSMISSION LINE

A pulse generator with a source impedance of 150-ohms is connected to a 400-meter length of 50-ohm coax. If the line is terminated in a 16.7-ohm load resistor

(a) what is the initial voltage and current on the line at $z=0$? at $t=0^+$?

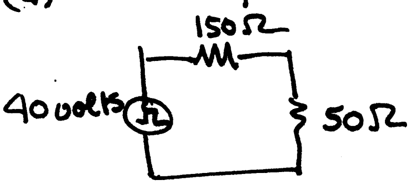
(b) what is the load voltage reflection coefficient? the load current reflection coefficient?

(c) Sketch $V(0,t)$ and $I(0,t)$ for $0 \le t \le 10\ \mu$seconds.



$$t = \frac{400\ m}{200\ m/\mu sec} = 2\ \mu sec.$$

(a) line impedance is seen as 50Ω initially



$$\therefore\ V = \frac{50}{50+150}\,40 = 10 \text{ volts.}$$

$$I = \frac{10 \text{ volts}}{50 \text{ ohms}} = 0.2 \text{ Amperes.}$$

(b) at $z=400$

$$\Gamma_L = \frac{R_L - R_c}{R_L + R_c} = \frac{16.7-50}{16.7+50} = -\frac{33.3}{66.7} \cong -\frac{1}{2} \quad (\text{voltage @ } z=400m)$$

$$\Gamma_I = -\Gamma_L = +\frac{1}{2} \quad (\text{current @ } z=400m)$$

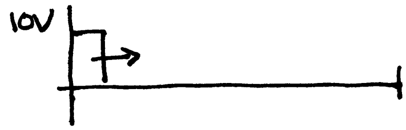
at $z=0$

$$\Gamma_s = \frac{R_s - R_c}{R_s + R_c} = \frac{150-50}{150+50} = \frac{100}{200} = +\frac{1}{2}$$

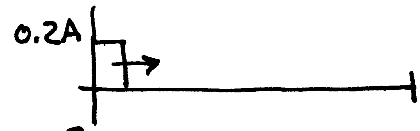
$$\Gamma_I = -\Gamma_s = -\frac{1}{2}.$$

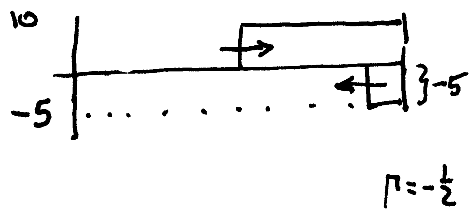
<u>How long is the pulse?</u>

voltage | current

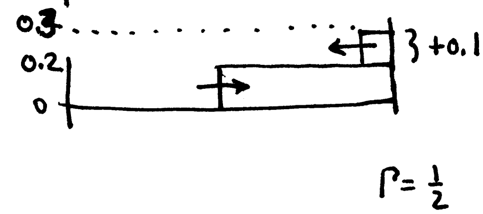


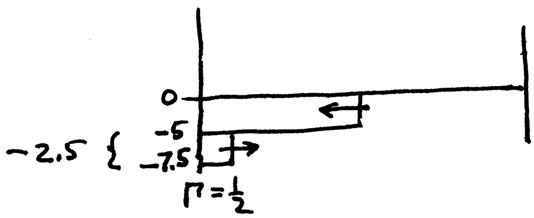
$t=0^+$



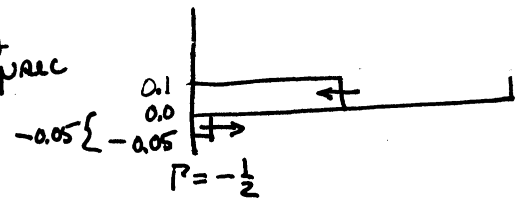


$t = 2\mu sec^+$

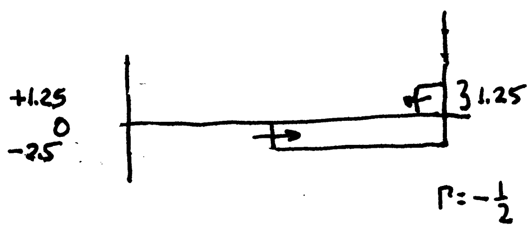


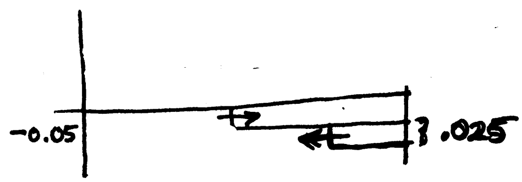


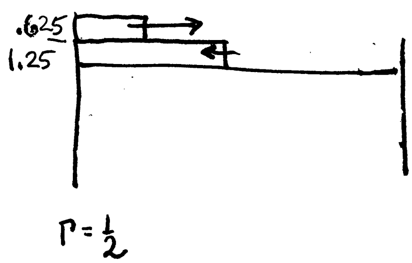
$t = 4^+ \mu sec$



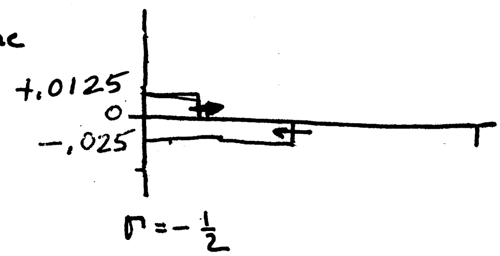
$t = 6^+ \mu sec.$



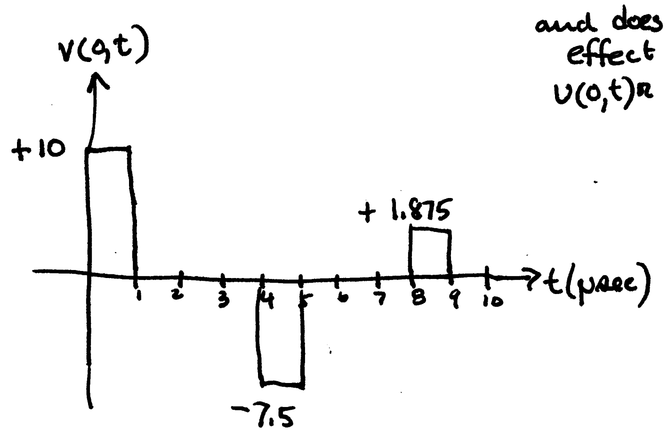


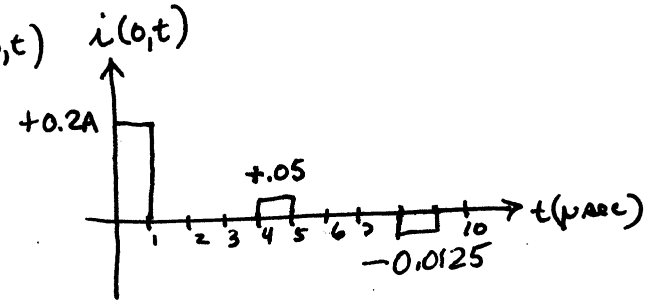


$t = 8^+ \mu sec$



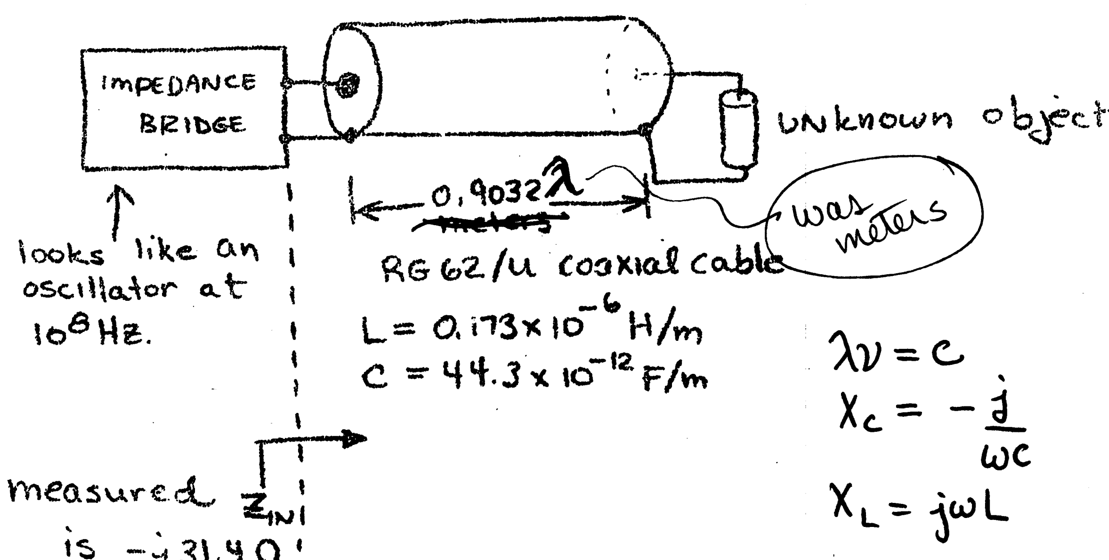
$t = 10^+ \mu sec$
at other end
and does not
effect
$v(0,t)$ & $i(0,t)$





3. SINUSOIDAL EXCITATION OF TRANSMISSION LINE

Mr. Casey Nerd is taking Circuits I. He finds a small cylindrical object on the floor which looks like the capacitor issued to the class for their laboratory. Being a conservative person Casey decides to play it safe and measure the object's value before putting it in his circuit. The digital LCR meter is not working but the lab assistant measures ~~its~~ the object's impedance on an impedance bridge. If the measurement is made as shown below and the lab assistant measures an impedance of $Z = -j31.4\Omega$, should Casey use the unknown object in his circuit? What is the unknown object, and what is its value?



$\lambda\nu = c$

$X_c = -\dfrac{j}{\omega C}$

$X_L = j\omega L$

$$R_c = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.173 \times 10^{-6}}{44.3 \times 10^{-12}}} = 62.5\Omega$$

$$u = \frac{1}{\sqrt{44.3 \times 10^{-12} \times 0.173 \times 10^{-6}}} = 0.36 \times 10^{9}\ \text{m/sec.}$$

$$\beta = \frac{\omega}{u} = \frac{2\pi \times 10^{8}/\text{sec}}{0.36 \times 10^{9}\ \text{m/sec}} = 1.739/\text{m.}$$

length
$$\lambda = \frac{c}{\nu} = \frac{3 \times 10^{8}\ \text{m/sec}}{10^{8}/\text{sec}} = 3\text{m.}$$

$$\beta l = 1.739 \times 0.9032 = 1.570\ \text{rad} \approx \frac{\pi}{2}.$$

was supposed to be 0.6571 rad. not 1.570 radians.

actually, if you use 0.9032λ then.

$$\beta l = \frac{2\pi}{\lambda}(0.9032\lambda) = 2\pi(0.9032)$$

$$\therefore \tan\beta l = -0.6963$$

$$Z_{in} = R_c \frac{Z_L + jR_c \tan\beta l}{R_c + jZ_L \tan\beta l}.$$

substituting into this expression

$$-j31.4 = 62.5 \frac{Z_L + j(62.5)(-.696)}{62.5 + jZ_L(-.696)}$$

and cross multiplying

$$-j1962.5 - 21.85Z_L = 62.5Z_L - j2718.75$$

$$j756.25 = 84.35Z_L$$

$$Z_L = j8.966\ \Omega$$

this is an inductor of value.

$$j8.966 = j\omega L.$$

$$L = \frac{8.966}{\omega} = \frac{8.966}{2\pi(10^8)} = 1.427\times10^{-8}\ \text{Henrys}.$$

$$= 0.143\ \text{nH}.$$

EEAP 210 ELECTROMAGNETIC FIELD THEORY

EXAM. NO. 5 (MAKE UP)

NAME : SOLUTIONS

INSTRUCTIONS : 1. DO ALL PROBLEMS (THERE ARE THREE)

2. WRITE NEATLY.

3. SHOW ALL YOUR WORK INDICATING WHERE FORMULAS YOU USE COME FROM.

GRADING

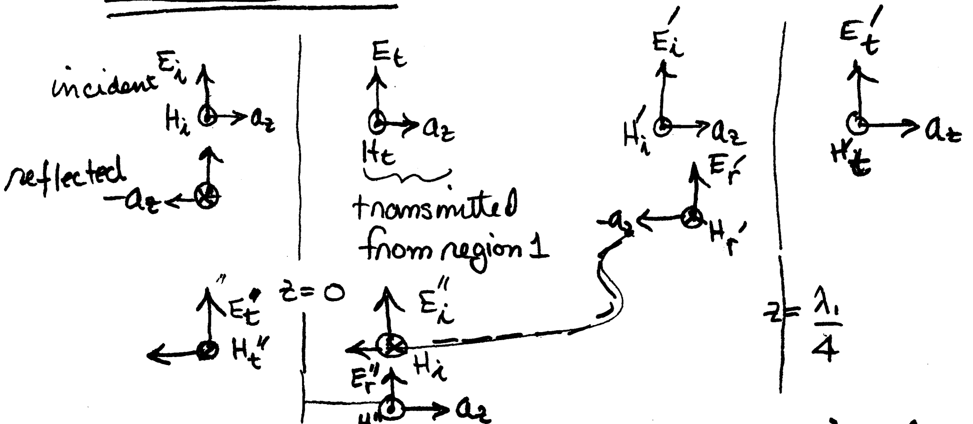
PROBLEM 1	
2	
3	
TOTAL	

ADVISORY GRADE

(a)

	region 1	region 2	region 3
$\beta = \omega\sqrt{\mu\epsilon}$	$\beta_1 = \omega\sqrt{\mu_0\epsilon_0}$	$\beta_2 = \omega\sqrt{4\mu_0\epsilon_0} = 2\omega\sqrt{\mu_0\epsilon_0} = 2\beta_1$	$\beta_3 = 3\beta_1$
$\lambda = \dfrac{2\pi}{\beta}$	$\lambda_1 = \dfrac{2\pi}{\beta_0}$	$\lambda_2 = \dfrac{2\pi}{\beta_2} = \dfrac{2\pi}{2\beta_1} = \dfrac{\lambda_1}{2}$	$\lambda_3 = \dfrac{\lambda_1}{3}$
$\eta = \sqrt{\dfrac{\mu}{\epsilon}}$	$\eta_1 = \sqrt{\dfrac{\mu_0}{\epsilon_0}}$	$\eta_2 = \sqrt{\dfrac{\mu_0}{4\epsilon_0}} = \dfrac{1}{2}\sqrt{\dfrac{\mu_0}{\epsilon_0}} = \dfrac{\eta_1}{2}$.	$\eta_3 = \dfrac{\eta_1}{3}$

(b) There must be.



Answer: By inspection the transmission at $z=0$ gives rise to a transmission and a reflection indicated by primes at $z = \lambda_1/4$. The reflection at $z = \frac{\lambda_1}{4}$ is incident upon the interface at $z=0$ giving rise to a reflection and transmission (indicated by the double primes) at $z=0$, which reflects a wave back to $z = \frac{\lambda_1}{4}$, and the process continues ad infinitum.

(c)

	region 1.	region 2	region 3
E:	$E_i e^{-j\beta_1 z} + E_r e^{+j\beta_1 z}$	$E_2^+ e^{-j\beta_2 z} + E_2^- e^{+j\beta_2 z}$	$E_3^+ e^{-j\beta_3 z}$
H:	$\dfrac{E_i e^{-j\beta_1 z} - E_r e^{+j\beta_1 z}}{\eta_1}$	$\dfrac{E_2^+ e^{-j\beta_2 z}}{\eta_2} - \dfrac{E_2^- e^{+j\beta_2 z}}{\eta_2}$.	$\dfrac{E_3^+ e^{-j\beta_3 z}}{\eta_3}$.

(d) tangential E continuous, tangential H continuous $\Big\}$ no current or charge on interfaces

(e)

at $z=0$

$$\boxed{\begin{aligned} E_i + E_r &= E_2^+ + E_2^- \\ \frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} &= \frac{E_2^+}{\eta_2} - \frac{E_2^-}{\eta_2} \end{aligned}}$$

at $z = \frac{\lambda_1}{4}$

$$E_2^+ e^{-j\beta_2\frac{\lambda_1}{4}} + E_2^- e^{+j\beta_2\frac{\lambda_1}{4}} = E_3^+ e^{-j\beta_3\frac{\lambda_1}{4}}$$

$$\frac{E_2^+ e^{-j\beta_2\frac{\lambda_1}{4}}}{\eta_2} - \frac{E_2^- e^{+j\beta_2\frac{\lambda_1}{4}}}{\eta_2} = \frac{E_3^+ e^{-j\beta_3\frac{\lambda_1}{4}}}{\eta_3}$$

we can simplify somewhat the expressions for $z = \frac{\lambda_1}{4}$.

as $\beta_2 = 2\beta_1 = 2\,\frac{2\pi}{\lambda_1}$

$$\beta_2 \frac{\lambda_1}{4} = 2 \cdot \frac{2\pi}{\cancel{\lambda_1}} \cdot \frac{\cancel{\lambda_1}}{4} = \pi$$

$$\beta_3 = 3\beta_1 = 3\,\frac{2\pi}{\lambda_1}$$

$$\beta_3 \frac{\lambda_1}{4} = 3\,\frac{\cancel{2}\pi}{\cancel{\lambda_1}} \cdot \frac{\cancel{\lambda_1}}{\cancel{4}_2} = \frac{3\pi}{2}$$

Then

$$e^{\pm j\pi} = \cos\pi \pm j\sin\pi = 1$$

$$e^{\pm j\pi\cdot\frac{3\pi}{2}} = \cos\frac{3\pi}{2} \pm j\sin\frac{3\pi}{2} = \mp j$$

and, re-writing the expressions for $z = \frac{\lambda_1}{4}$.

$$\boxed{\begin{aligned} E_2^+ + E_2^- &= +jE_3^+ \\ \frac{E_2^+}{\eta_2} - \frac{E_2^-}{\eta_2} &= +j\,\frac{E_3}{\eta_3} \end{aligned}}$$

re-writing

$$1 + \frac{E_r}{E_i} = \frac{E_2^+ + E_2^-}{E_i}$$

define $\Gamma_1 = \dfrac{E_r}{E_i}$

$T_1 = \dfrac{E_2^+ + E_2^-}{E_i}$

$$\frac{1}{\eta_1} - \frac{1}{\eta_2}\frac{E_r}{E_i} = \frac{E_2^+ - E_2^-}{\eta_2 \, E_i}$$

$$1 + \frac{E_2^-}{E_2^+} = \frac{j E_3^+}{E_2^+}$$

define $\Gamma_2 = \dfrac{E_2^-}{E_2^+}$

$$\frac{1}{\eta_2} - \frac{E_2^-}{\eta_2 E_2^+} = \frac{j}{\eta_3}\frac{E_3^+}{E_2^+}$$

$T_2 = \dfrac{E_3^+}{E_2^+}$

and solve equations (4 equations in 4 unknowns,

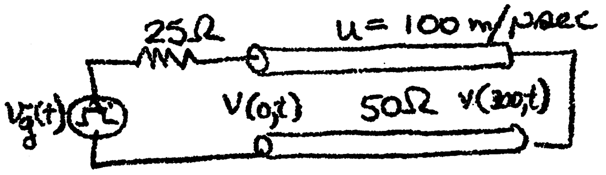
$\dfrac{E_r}{E_i}$, $\dfrac{E_2^+}{E_i}$, $\dfrac{E_2^-}{E_i}$ and $\dfrac{E_3^+}{E_i}$

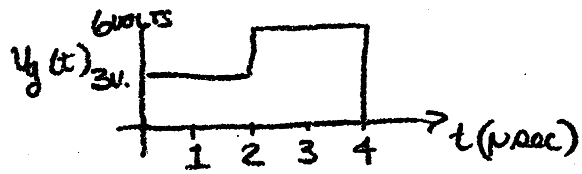
2. PULSED TRANSMISSION LINES

A pulse generator with a source impedance of 25 ohms is connected to a transmission line of characteristic impedance 50Ω. The line is 300 meters long and terminated in a short.

(a) what are the source and load reflection coefficients.

(b) sketch V(0,t) for $0 \le t \le 10$ μseconds for the pulse excitation shown below.

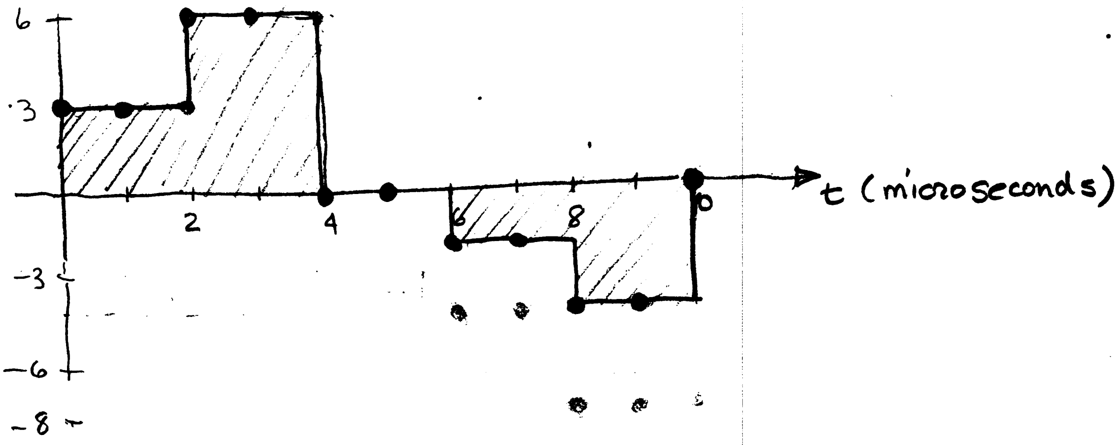




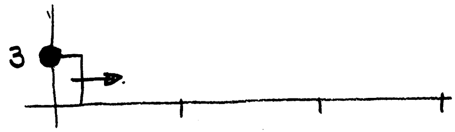
$$V_g = \begin{cases} 3 \text{ VOLTS} & 0<t<2\ \mu sec \\ 6 \text{ volts} & 2<t<4\ \mu sec. \\ 0 & t>4\ \mu sec. \end{cases}$$

(a) $\Gamma_L = \dfrac{Z_L - R_c}{Z_L + R_c} = -\dfrac{R_c}{R_c} = -1$

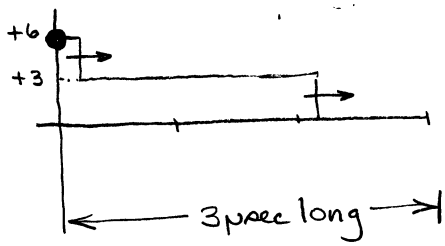
$\Gamma_s = \dfrac{Z_s - R_c}{Z_s + R_c} = \dfrac{25-50}{25+50} = \dfrac{-25}{75} = -\dfrac{1}{3}$



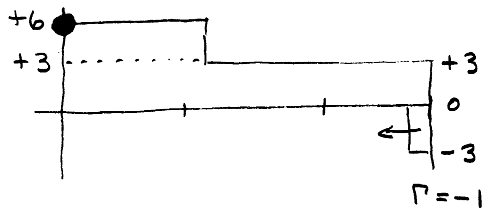
t=0+



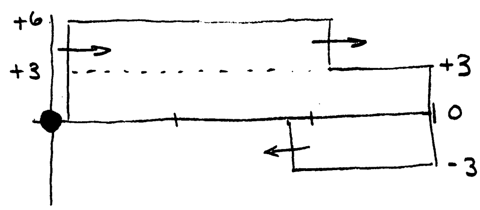
t=2+



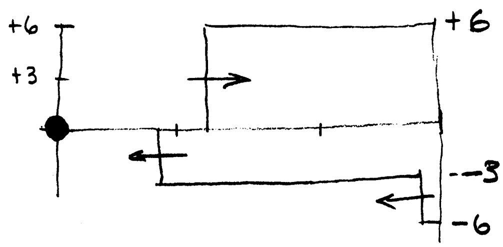
t=3+



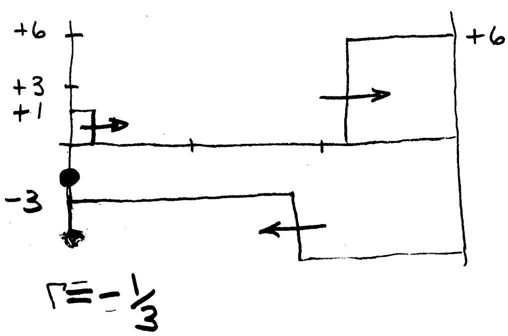
t=4+



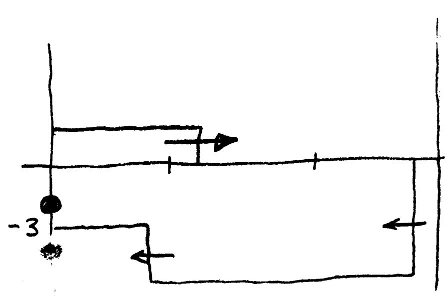
t=5+



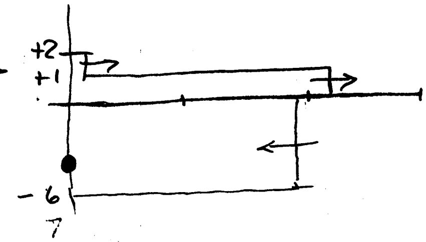
t=6+



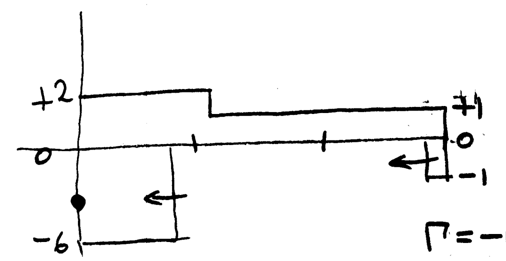
t=7+



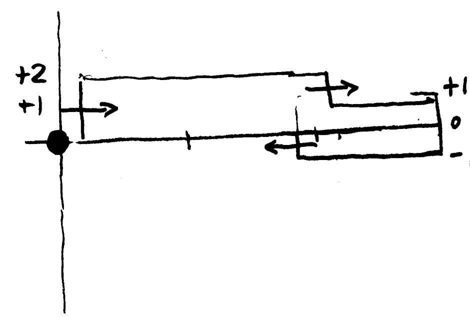
t=8+



t=9+



t=10+

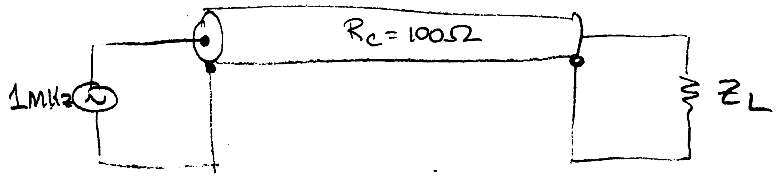


Note: heavy dots here correspond to heavy dots on previous page.

3. SINUSOIDAL TRANSMISSION LINES

A long, $R_c = 100\,\Omega$, lossless transmission line is terminated in an unknown impedance and operated at a frequency of 1 MHz. The input impedance is measured as $10 - j50\,\Omega$.

If the unknown load impedance is removed, i.e. the line now is open at the load, and the input impedance is now $-j75\,\Omega$, determine the original load impedance. Hint: Let Z_L (unknown) $= R + jX$, you don't need exponentials for this problem.



$$Z_{IN} = R_c \frac{Z_L + jR_c \tan\beta l}{R_c + jZ_L \tan\beta l}$$

both $\tan\beta l$ and Z_L are unknown, but the problem gives us two equations in two unknowns.

let

$$10 - j50 = 100\left(\frac{Z_L + j100 \tan\beta l}{100 + jZ_L \tan\beta l}\right) \qquad \cancel{-j}75 = \frac{100(\cancel{-j})}{\tan\beta l}$$

$$\tan\beta l = \frac{100}{75} = 1.333$$

$$10 - j50 = \frac{100\,(Z_L + j133)}{100 + j1.33 Z_L}$$

$$(10 - j50)(100 + j1.33 Z_L) = 100(Z_L + j133)$$

$$1000 - j5000 + 66.50 Z_L + j13.30 Z_L = 100 Z_L + j13300.$$

let $Z_L = R + jX$

$$1000 - j5000 + 66.50R + j66.50X + j13.30R - 13.30X = 100R + j100X + j13300$$

real: $1000 + 66.50R - 13.30X = 100R$

imaginary: $-5000 + 66.50X + 13.30R = 100X + 13300$

$$-33.5R - 13.30X = -1000$$
$$13.30R - 33.5X = 18300$$

$$-33.5R - 13.3X = -1000$$

$$\frac{33.5}{13.3}\ (13.3R - 33.5X = 18300$$

$$-33.5R - 13.3X = -1000.$$

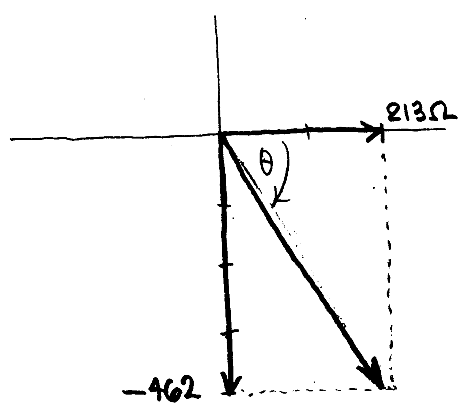
$$33.5R - 84.38X = 46093.98$$

$$-97.68X = 45093.98$$

$$X = -461.7$$

$$13.3R + 15466.95 = 18300$$

$$R = \frac{2833.05}{13.3} = 213.0$$



$$\sqrt{(213)^2 + (462)^2}$$

$$\approx 508.7 \angle \tan^{-1}\left(\frac{-462}{213}\right)$$

by another method

$$Z_{IN} = R_c \frac{Z_L + j R_c \tan \beta l}{R_c + j Z_L \tan \beta l}.$$

$$Z_{IN} R_c + j Z_{IN} Z_L \tan \beta l = R_c Z_L + j R_c^2 \tan \beta l.$$

$$Z_{IN} R_c - j R_c^2 \tan \beta l = R_c Z_L - j Z_{IN} Z_L \tan \beta l$$

$$Z_L = \frac{R_c (Z_{IN} - j R_c \tan \beta l)}{R_c - j Z_{IN} \tan \beta l}$$

$$= 100 \left(\frac{10 - j50 - j100\,(1.33)}{100 - j(10 - j50)(1.33)} \right)$$

$$= 100 \frac{(10 - j50 - j133)}{100 - j(13.3 - j66.5)}$$

$$= 100 \frac{10 - j183}{100 - j13.3 - 66.4} = 100 \frac{10 - j183}{33.6 - j13.3} \cdot \frac{33.6 + j13.3}{33.6 + j13.3}$$

$$= 100 \frac{336 + 2433.9 - j6148.8 + j133}{1128.96 + 176.89}$$

$$= 100 \frac{2769.9 - j6015.8}{1305.85}$$

$$= 212 - j460.7$$

Correct.

EEAP 210 ELECTROMAGNETIC FIELD THEORY
MAY 5, 1983

FINAL EXAM

NAME: SOLUTIONS

Instructions:

1. Do ALL problems - There are five (5)
2. write neatly
3. Show all your work - indicate where your results came from

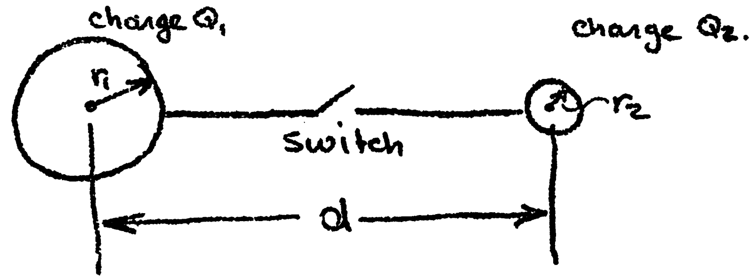
GRADING FOR EXAM

ELECTROSTATICS		4 POINTS
PLANE WAVES		4 POINTS
PULSES		5 POINTS
CAPACITANCE		5 POINTS
MAGNETIC FIELDS		5 POINTS
TOTAL:		23 POINTS

EXAM GRADE:

COURSE GRADE:

Electrostatics



Two conducting spheres with radii r_1 and r_2 and surface charges (total) Q_1 and Q_2 respectively have their centers a distance d apart as shown above. The charges are uniformly distributed. Neglecting any interaction between the spheres,

(a) What is the potential on each sphere in terms of Q and r. Find the field first using Gauss' Law and then find Φ.

(b) If a switch is used to short the two spheres together, what happens to the potential at each sphere's surface?

(c) What are the new electric fields? Hint: $Q_1 + Q_2 = \text{constant}$.

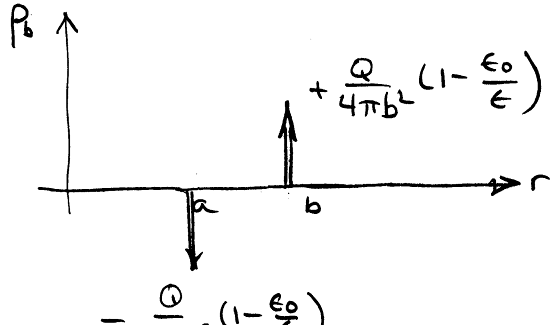
Yes, there is a bound charge on the interior and exterior surfaces of the dielectric sphere. This charge appears as the shell atoms align themselves with the applied field. In the interior of the shell the net macroscopic charge density due to the electric dipoles is zero; however, on the surfaces there is no net cancelation and a bound charge appears on the surface.

The charge density can be gotten from $\rho_b = -\underline{\nabla} \cdot \underline{P}$

$$\rho_b = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\frac{Q}{4\pi \cancel{r^2}}\left(1-\frac{\epsilon_0}{\epsilon}\right)\right)$$

$$= -\frac{Q}{4\pi r^2}\frac{\partial}{\partial r}\left[1-\frac{\epsilon_0}{\epsilon}\right]$$

$$= -\frac{Q}{4\pi r^2}\left[\left(1-\frac{\epsilon_0}{\epsilon}\right)\delta(r-a) - \left(1-\frac{\epsilon_0}{\epsilon}\right)\delta(r-b)\right]$$

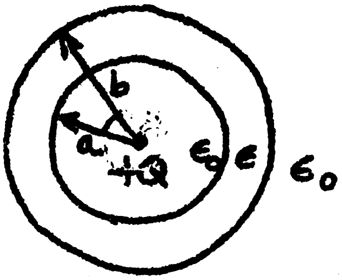


Note that this is exactly the discontinuity in the E field.

There is no free charge on the interface as.

$$\rho_f = D_2 - D_1$$

Electrostatics



A <u>positive</u> point charge is surrounded by a spherical shell of dielectric material as shown above.

(a) What are $\underline{E}$, $\underline{D}$ and $\underline{P}$ everywhere? Plot them as functions of r.

(b) Is there any charge on the dielectric shell? If so, where, how much and from where does it come?

Gauss' Law: $\oint \underline{D}\cdot d\underline{s} = Q.$

$$\epsilon E \cdot 4\pi r^2 = +Q$$

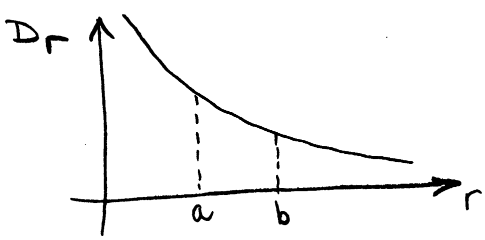
all $\underline{a_r}$ directed

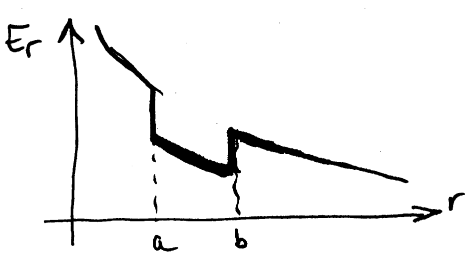
$$E = \frac{Q}{4\pi\epsilon r^2}$$

$$D = \frac{Q}{4\pi r^2}$$

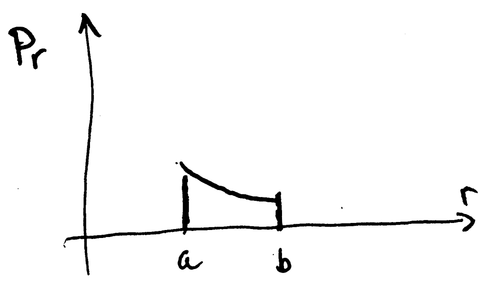
$$D = \epsilon_0 E + P$$

$$\therefore\ P = D - \epsilon_0 E$$





-0.5 each



$$P = \frac{Q}{4\pi r^2} - \epsilon_0 \frac{Q}{4\pi\epsilon r^2}$$

$$= \frac{Q}{4\pi r^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right)$$

$$= \begin{cases} \dfrac{Q}{4\pi r^2}\left(1 - \dfrac{\epsilon_0}{\epsilon}\right) & a < r < b \\ 0 & \text{elsewhere.} \end{cases}$$

(a). Since they are far apart.

gauss law for each.

$$\int \mathbf{D} \cdot dS = Q.$$

$$\epsilon_0 E_r \cdot 4\pi r^2 = Q_1$$

$$E_r = \frac{Q_1}{4\pi\epsilon_0 r^2}.$$

$$\phi_1 = -\int E \cdot dl = -\int_{-\infty}^{r_1} \frac{Q_1}{4\pi\epsilon_0} r^{-2} dr = \frac{-Q_1}{4\pi\epsilon_0} -r^{-1}\Big|_{-\infty}^{r_1}$$

$$= -\frac{Q_1}{4\pi\epsilon_0}\left[-\frac{1}{\infty} + \frac{1}{r}\right] = \frac{-Q_1}{4\pi\epsilon_0 r_1}$$

like wise for the second sphere $\phi_2 = \frac{Q_2}{4\pi\epsilon_0 r_2}$

(b) potentials must be equal.

r's cannot change so Q's must change call the new charges Q_1' and Q_2'

$$\phi_1 = \phi_2$$

$$\frac{Q_1'}{4\pi\epsilon_0 r_1} = \frac{Q_2'}{4\pi\epsilon_0 r_2} \quad \text{and } Q_1' + Q_2' = Q_1 + Q_2$$

$$\therefore Q_2' = (Q_1 + Q_2) - Q_1'$$

$$\frac{Q_1'}{\cancel{4\pi\epsilon_0} r_1} = \frac{(Q_1 + Q_2) - Q_1'}{\cancel{4\pi\epsilon_0} r_2}$$

$$Q_1'\left[\frac{1}{r_1} + \frac{1}{r_2}\right] = \frac{Q_1 + Q_2}{r_2}$$

$$Q_1' = \frac{\frac{Q_1 + Q_2}{\cancel{r_2}}}{\frac{r_2 + r_1}{r_1 \cancel{r_2}}} = \left(\frac{r_1}{r_1 + r_2}\right)(Q_1 + Q_2)$$

$$Q_2' = (Q_1+Q_2) - \left(\frac{r_1}{r_1+r_2}\right)(Q_1+Q_2) = \frac{r_2}{r_1+r_2}(Q_1+Q_2)$$

(c) as in (b) only the Q's changed

$$E_1'(r) = \frac{Q_1'}{4\pi\epsilon_0 r^2} = \left(\frac{r_1}{r_1+r_2}\right)\frac{Q_1+Q_2}{4\pi\epsilon_0 r^2}$$

$$E_2'(r) = \frac{Q_2'}{4\pi\epsilon_0 r^2} = \left(\frac{r_2}{r_1+r_2}\right)\frac{Q_1+Q_2}{4\pi\epsilon_0 r^2}$$

r is measured radially away from each center.

PLANE WAVES

The electric field intensity of a uniform plane wave propagating in the +z direction in sea water is

$$\underline{E} = \underline{a_x}\ 100 \cos(10^7 \pi t) \text{ volts/meter}$$

at $z=0$. The propteries of sea water are $\epsilon = 80\epsilon_0$, $\mu = \mu_0$ and $\sigma = 4 /\text{ohm-m}$

(a) What is the linear frequency f of this wave?

(b) Is sea water a conductor or a dielectric? Why?

(c) Find α and β for sea water?

(d) What is the complex impedance of sea water?

(e) Write phasor expressions for the $\underline{E}$ and $\underline{H}$ fields everywhere in the sea water.

(f) Determine the distance d at which the amplitude of the electric field decreases to 1% of its value at $z=0$. (Just set up if your calculator does not have the appropriate functions)

(a) $2\pi f = \omega = 10^7 \pi$

$2f = 10^7$

$f = 0.5 \times 10^7$ Hz.

(b) compare magnitude of conduction and dielectric current.

$$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) = j^2\omega\mu\omega\epsilon\left(\frac{\sigma}{j\omega\epsilon} + 1\right)$$

$$= -\omega^2\mu\epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)$$

$$\frac{\sigma}{\omega\epsilon} = \frac{4}{\cancel{\pi}10^7\, 80\left(\frac{1}{36\cancel{\pi}} \times 10^{-9}\right)}$$

$$= \frac{4 \cdot 36}{80 \times 10^{-2}} = 1.8 \times 10^2 = 180$$

$\therefore$ conduction current dominates and this is a good conductor.

(c) if $\quad j\frac{\sigma}{\omega\epsilon} >> 1 \quad$ then

$$\gamma^2 \approx + \quad \omega^{\cancel{2}} \mu \cancel{\epsilon} \frac{j\sigma}{\cancel{\omega}\cancel{\epsilon}} = j\,\omega\mu\sigma$$

$$\gamma = \sqrt{j}\sqrt{\omega\mu\sigma} = \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\therefore \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{\pi\,\cancel{10^7}\cdot 4\pi\times\cancel{10^{-7}}\cdot 4}{2}}$$

$$= \frac{\pi 4}{\sqrt{2}} = 8.886/m.$$

(d) $$\hat{\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \qquad \text{but } \sigma >> j\omega\epsilon$$

$$\cong \sqrt{\frac{j\omega\mu}{\sigma}} \approx \frac{1+j}{\sqrt{2}}\sqrt{\frac{\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}}$$

$$\sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{\pi\,\cancel{10^7}\;\cancel{4}\pi\times\cancel{10^{-7}}}{2\cdot\cancel{4}}} = \frac{\pi}{\sqrt{2}} = 2.221$$

$$\hat{\eta} = (1+j)\,2.221\ \Omega.$$

$1.77\angle 22.5°$.
$\underline{9.8\angle 4.7°}$

(e) $$\hat{E}(z,t) = E_0 e^{-\gamma z} = E_0 e^{-\alpha z} e^{-j\beta z}$$

but at $z=0$ $\quad \hat{E}(z,t) = E_0$

and $\mathcal{E}(z,t) = \mathrm{Re}\left(E_0 e^{j\omega t}\right) = E_0 \cos\omega t$.

we have $E_0 = 100$ volts/meter

$$\therefore \quad \hat{E}(z,t) = 100\, e^{-8.886z}\, e^{-j8.886z}$$

$$\hat{H}(z,t) = \frac{\hat{E}(z,t)}{\eta} = \frac{100}{2.221(1+j)}\cdot\frac{1-j}{1-j}\, e^{-8.886z}\, e^{-j8.886z}$$

$$= 22.5\, e^{-8.886z - j8.886z}.$$

(f) as a function of z the amplitude goes as.

$$E(z,t) = \underbrace{100\, e^{-8.886z}}_{\text{this is the amplitude part of the expression}} e^{-j8.886z}$$

$$E(d,t) = \underbrace{100\, e^{-8.886d}}_{\text{this equals 1 at } z=d} e^{-j8.886d}$$

since at $z=0$ this equal 100

$$\therefore \quad 1 = 100\, e^{-8.886d}.$$

$$.01 = e^{-8.886d}$$

$$d = \frac{-\ln(.01)}{8.886}$$

$$= 0.518 \text{ meters}$$

if the penetration (skin depth) is so large

<u>can</u> this be a <u>good</u> conductor?

PULSED TRANSMISSION LINE

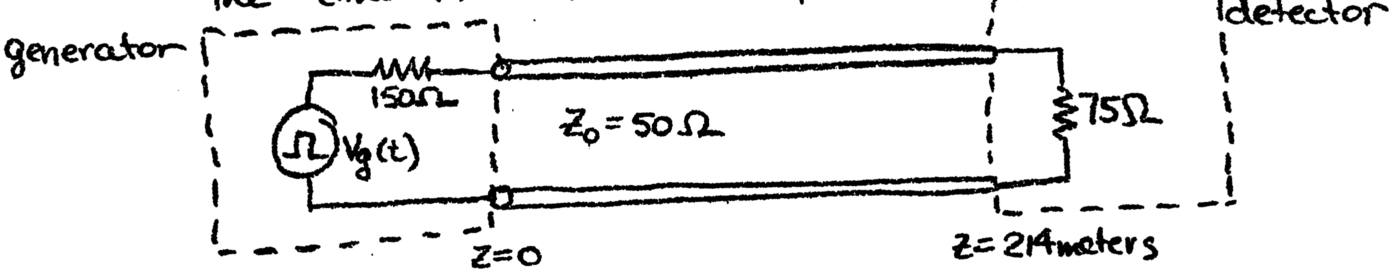
A pulse generator having an open circuit voltage of 40 volts and an internal resistance of 150Ω produces pulses of 0.1 μsec duration with a period of 3 μsec as shown below. The generator is connected to a 214 meter length of RG58A/U coaxial cable ($Z_c = 50\Omega$, $c = 93.5\times10^{-12}$ F/m). The other end of the system is a pulse detector having an input impedance of 75 ohms. The owner of the system finds that, after the generator is turned on, the time between the pulses received by the detector is <u>NOT</u> 3 μsec. What is happening?

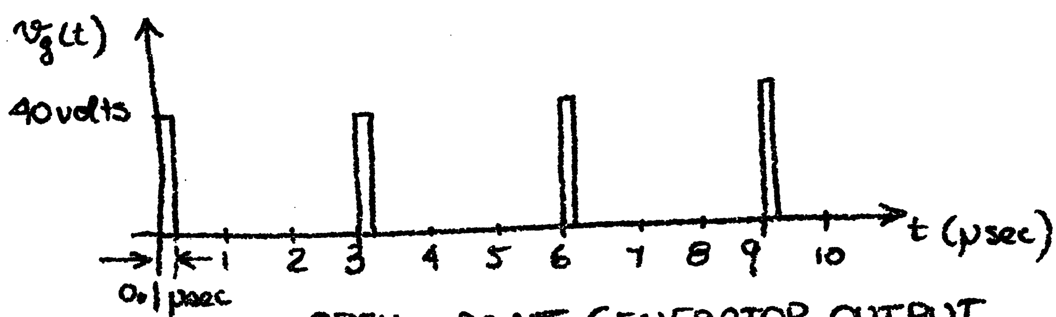
(a) Calculate u, the propagation velocity on the cable.

$$\epsilon_0 = \frac{1}{36\pi}\times10^{-9}\ \text{F/m},\quad \mu_0 = 4\pi\times10^{-7}\ \text{H/m}.$$

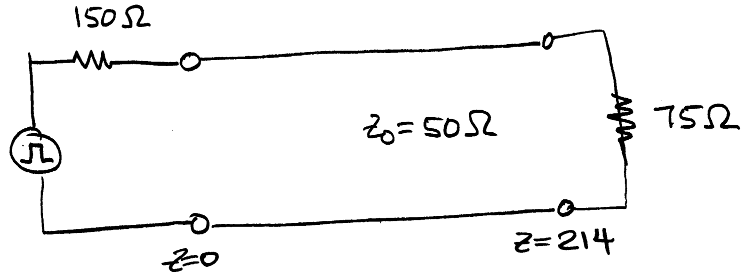
(b) What are the reflection (voltage) coefficients at each end of the cable.

(c) Plot the detector voltage (V(214 meters, t)), as a function of time for $0 \le t \le 14$ μsec. Specifically indicate the voltage of each pulse seen by the detector. What is the time interval between pulses?





OPEN CIRCUIT GENERATOR OUTPUT



(a) $R_c = \sqrt{\frac{L}{C}} \quad \therefore \quad R_c^2 C = L$

$$u = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{R_c^2 C \cdot C}} = \frac{1}{R_c C} = \frac{1}{(50)(93.5 \times 10^{-12})}$$

$$= 2.139 \times 10^{-4} \times 10^{+12} = 2.139 \times 10^{8} \text{ m/sec}$$

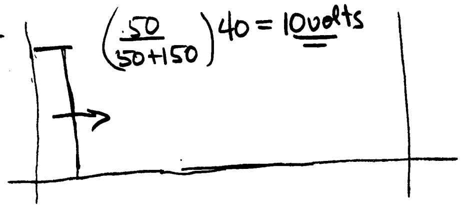
(b)

at $z = 214$m.

$$\Gamma_L = \frac{R_L - R_c}{R_L + R_c} = \frac{75-50}{75+50} = \frac{25}{125} = \frac{1}{5}$$

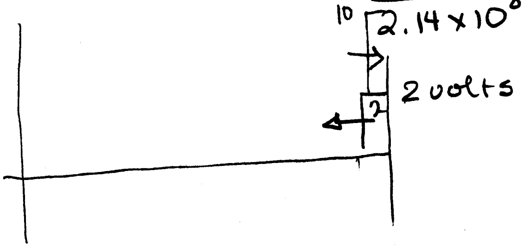
$$\Gamma_s = \frac{R_s - R_c}{R_L + R_c} = \frac{150-50}{150+50} = \frac{100}{200} = \frac{1}{2}$$

(c). $t=0^+$ $\left(\frac{50}{50+150}\right)40 = \underline{10 \text{ volts}}$

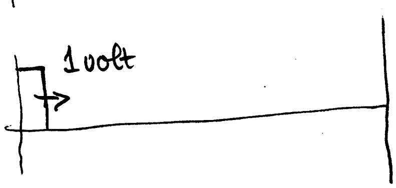


$$\frac{214\text{m}}{2.14 \times 10^8 \text{ m/sec}} = 10^2 \times 10^{-8} = 1.0\mu\text{sec.}$$

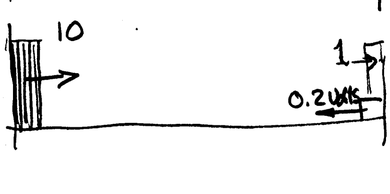
$t = 1^+$

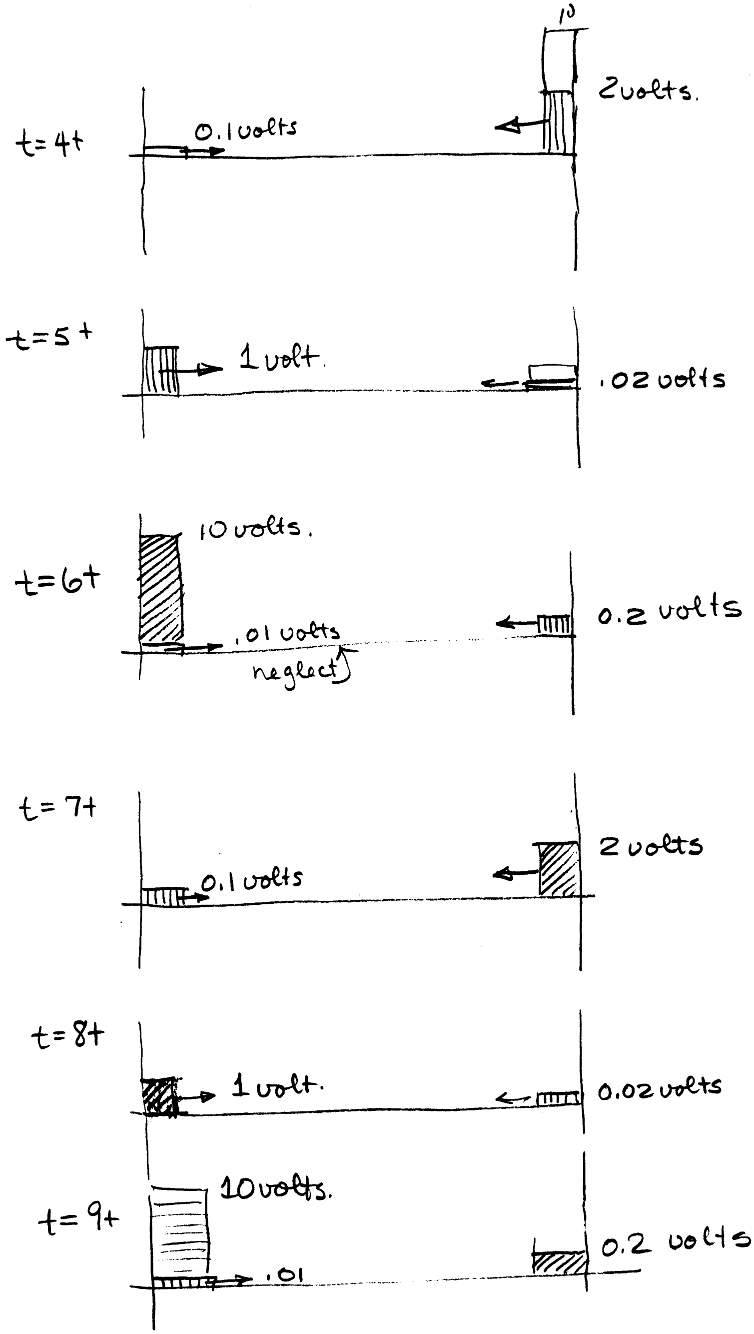


$t = 2^+$

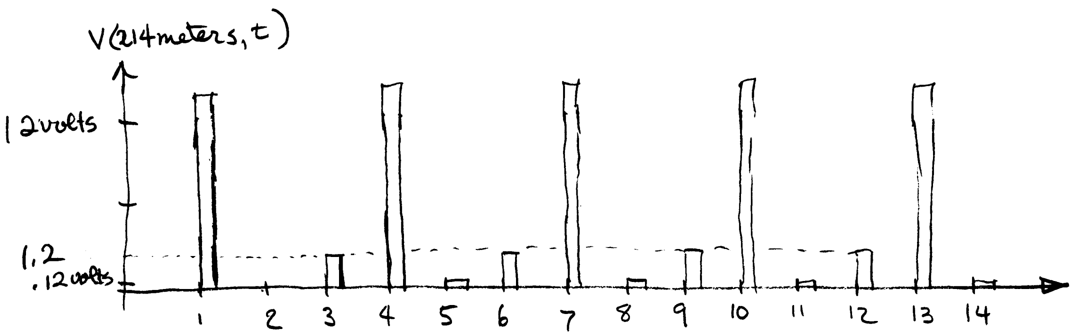


$t = 3^+$

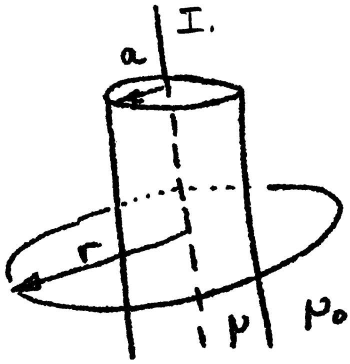



10
t=4t
0.1 volts
2 volts.
t=5+
1 volt.
.02 volts
t=6t
10 volts.
.01 volts
neglect
0.2 volts
t= 7t
0.1 volts
2 volts
t=8t
1 volt.
0.02 volts
t= 9t
10 volts.
.01
0.2 volts

etc.


V(214 meters, t)
12 volts
1.2
.12 volts
1
2
3
4
5
6
7
8
9
10
11
12
13
14

Magneto Statics



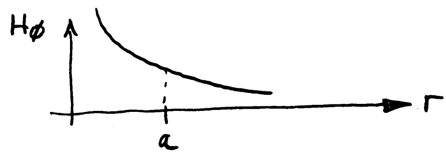
A line current I passes down the center of an infinitely long cyclinder of radius a and permeability μ. The cyclinder is surrounded by free space.

(a) Find $\underline{B}$, $\underline{H}$ and $\underline{m}$ everywhere. Plot your results as a function of r.

(b) What is the magnetization current $\underline{J}_m = \underline{\nabla} \times \underline{m}$. Plot $\underline{J}_m$ as a function of r.

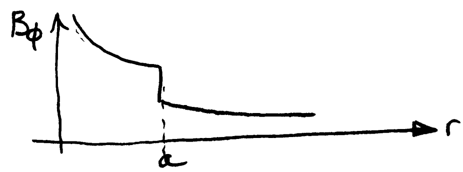
(c) What is a magnetization current? Discuss it in terms of the dipole moments in the material.

(a) $$\oint H \cdot dl = H_\phi 2\pi r = I \qquad \therefore H_\phi = \frac{I}{2\pi r}$$



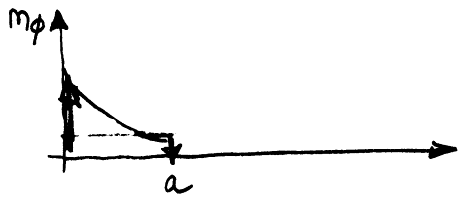
$$\therefore \quad H_\phi = \frac{I}{2\pi r} \qquad r > 0$$

$$B_\phi = \begin{cases} \dfrac{\mu I}{2\pi r} & 0 < r < a \\[2mm] \dfrac{\mu_0 I}{2\pi r} & a < r \end{cases}$$



$$\underline{B} = \mu_0 (\underline{H} + \underline{m})$$

$$\frac{\underline{B}}{\mu_0} - \underline{H} = \underline{M}$$



$$\underline{m} = \frac{\frac{\mu I}{2\pi r}}{\mu_0} - \frac{I}{2\pi r} = \left(\frac{\mu}{\mu_0} - 1\right) \frac{I}{2\pi r}$$

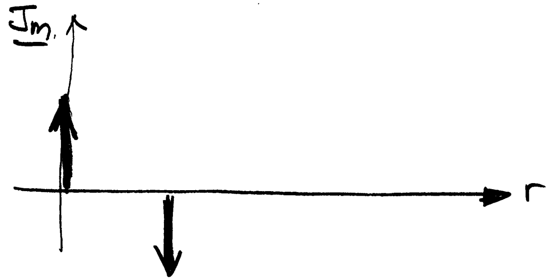
(b)

$$\underline{J_m} = \underline{\nabla} \times \underline{m}$$

$$= \underline{a_z}\, \frac{1}{r} \frac{\partial}{\partial r} (r\, m_\phi)$$

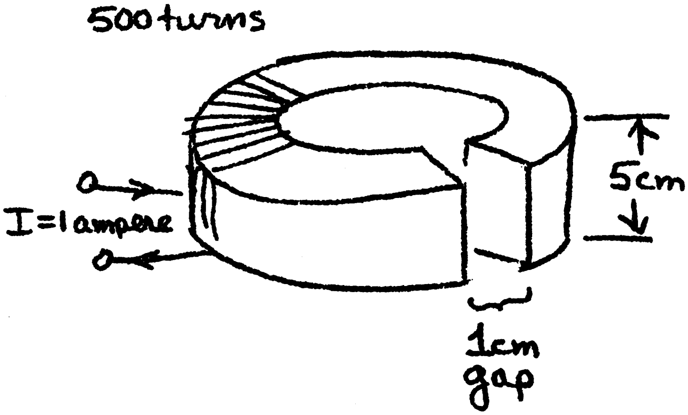
$$= \underline{a_z}\, \frac{1}{r} \frac{\partial}{\partial r} \left(\left(\frac{\mu}{\mu_0} - 1\right) \frac{I}{2\pi} \right)$$

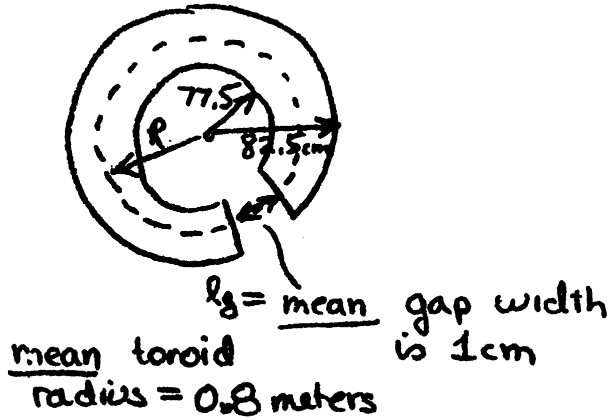
$$= \underline{a_z} \left[\left(\frac{\mu}{\mu_0} - 1\right) \frac{I}{2\pi} \delta(r) - \left(\frac{\mu}{\mu_0} - 1\right) \frac{I}{2\pi a} \delta(r-a) \right]$$



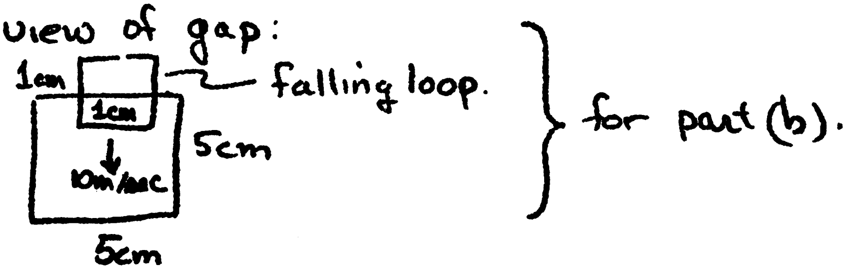
(c) Magnetization current is due to the motion of bound charges in materials. The only place where there is a <u>net</u> motion of bound charges is at the surface of magnetic materials; hence, magnetic materials have magnetization currents on their surfaces, — not in their interiors.

MAGNETIC FIELDS





side view of gap:



A toroidal iron core of relative permeability $3000\mu_0$ has a mean radius $R = 0.8$ meters and a 5×5 cm square cross section. An air gap of <u>mean</u> width $l_g = 1$ cm exists. A d.c. current $I = 1$ ampere flows in a 500 turn winding around the toroid.

(a) Find the magnetic field $\underline{B}$ in the gap.

(b) If a square loop of wire of dimensions 1×1 cm falls through the gap at a constant velocity of $v = 10$ m/sec what is the induced voltage in the loop as a function of time? Plot your results. You may assume that the loop falls straight, does not rotate or move sideways out of the field, and that gravity can be neglected.

Magnetic Fields

(a) There are two ways to find the magnetic field B in the gap. The first is to use the magnetic circuit approach. The second is to use Ampère's Law. As Ampère's Law is used to derive (develop) the magnetic circuit approach, both yield identical results under the same approximations. We will use the magnetic circuit approach.

The first point is to recall that, for magnetic circuits, one uses the mean path. This puts the contour as far away as possible from edge effects and fringing fields.

For the given toroid mean radius = 0.8 meters
mean gap = .01 meters

$NI = (500)(1 \text{ ampere}) = 500$ A-t

The cross-sectional area is

$$(.05\text{m tall})(.825 - .775\text{m}) = 2.5 \times 10^{-3}\ \text{m}^2$$

The reluctance of the gap.

$$\mathcal{R}_g = \frac{l}{\mu_0 A} = \frac{10^{-2}}{(4\pi\times10^{-7})(2.5\times10^{-3})}$$

$$= 3.18\times10^{-2}\times10^{8} = 3.18\times10^{6}\ \frac{\text{A-t}}{\text{Wb}}$$

$$\mathcal{R}_{iron} = \frac{l}{\mu A} = \frac{2\pi(.8) - .01}{3\times10^{3}\times 4\pi\times10^{-7}\times 2.5\times10^{-3}}$$

$$\cong .0533\times10^{7} = 5.33\times10^{5}\ \frac{\text{A-t}}{\text{Wb.}}$$

∴ ~~It is questionable if~~

∴ The reluctance of the iron cannot be neglected.

$$\mathcal{F} = NI = 500 \text{ A-t} = \Phi(\mathcal{R}_{gap} + \mathcal{R}_{iron})$$

$$\therefore \Phi = \frac{500 \text{ A-t}}{\mathcal{R}_{gap} + \mathcal{R}_{iron}}$$

$$= \frac{500 \text{ A-t}}{31.8 \times 10^5 + 5.33 \times 10^5 \ \frac{\text{A-t}}{\text{Wb}}}$$

$$= 13.46 \times 10^{-5} \text{ Wb}.$$

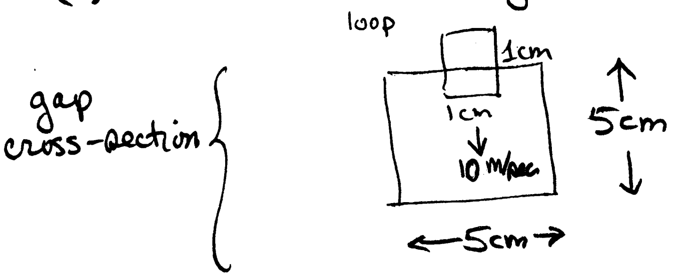
$$= 1.346 \times 10^{-4} \text{ Wb}.$$

Since $\Phi = BA$

$$B = \frac{\Phi}{A} = \frac{1.346 \times 10^{-4}}{.2.5 \times 10^{-3}} = .5384 \times 10^{-1} = .05384 \ \frac{\text{Wb}}{\text{m}^2}$$

which is assumed to be uniform over the gap.

(b) This is the easy part of the problem.



$S = vt$

what is the flux through the loop as a function of time.

The loop will just enter the gap at $t = 0^+$

It will be entirely in the gap when $t = \frac{1\text{cm}}{1000 \frac{\text{cm}}{\text{sec}}} = 1 \text{ msec}$

It will remain entirely in the gap for 4. msec. (think about it)

And will begin leaving the gap at $1 + 4\text{msec} = 5 \text{ msec}$.

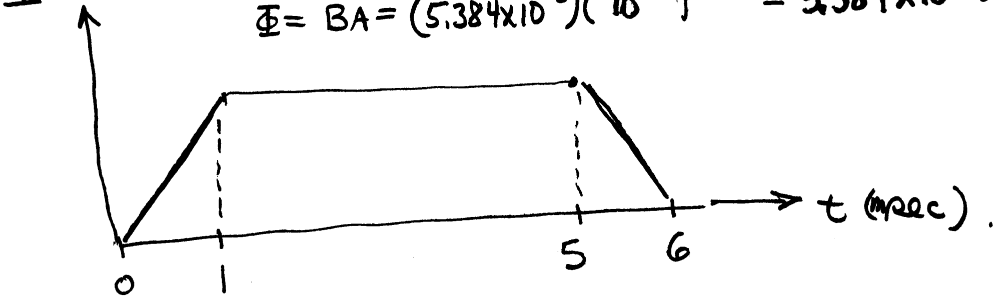
and will take 1 msec to leave.

m. fields

constant flux

area of loop

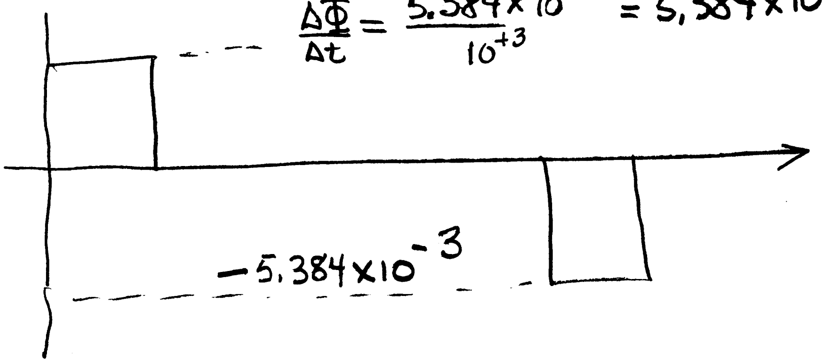
$\Phi = BA = (5.384\times10^{-2})(10^{-4}) = 5.384\times10^{-6}$ wb.



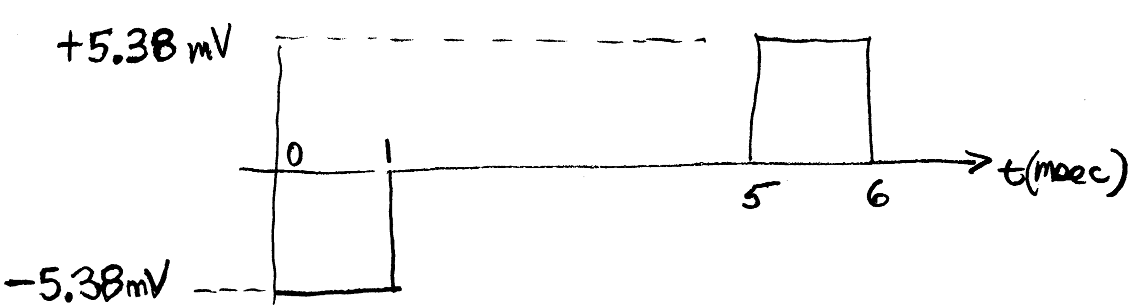
Because the velocity is linear, the flux through the loop increases linearly to its maximum.

As $v = -\dfrac{d\Phi}{dt}$. From the above plot we get for $\dfrac{d\Phi}{dt}$

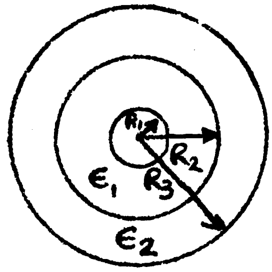
$$\frac{\Delta\Phi}{\Delta t} = \frac{5.384\times10^{-6}}{10^{+3}} = 5,384\times10^{-3}$$



Converting to voltage.



Capacitance



A capacitor is formed from two cyclindrical conductors of length l and radii R_1 and R_3 respectively.
The permittivity of the material between the electrodes is ϵ_1 for $R_1 \leq r < R_2$ and ϵ_2 for $R_2 \leq r < R_3$.
The inner conductor is at zero volts, the outer at $+V_0$ volts.

3 → for this (a) Find the potential and electric field inside the capacitor. Hint: Use Laplace's Equation

2 for this → (b) What is the capacitance of this capacitor?

NOTE: THIS IS A HARD PROBLEM:

From Laplace's Equation $\quad \nabla^2 \phi = 0$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0$$

$$r\frac{d\phi}{dr} = c_1$$

$$d\phi = c_1 \frac{dr}{r}$$

1 point to here.

$$\therefore \phi(r) = c_1(\ln r + c_2)$$

For those who inquired about the constants. The constants are determined by the voltage boundary conditions...

$$\phi(R_1) = 0 = c_1(\ln R_1 + c_2.) \qquad \text{in region } 1$$

$$\therefore c_2 = -\ln R_1$$

$$\phi(r) = c_1(\ln r - \ln R_1)$$

ϕ is continuous — one of the reasons it is so useful. so I don't need to worry about the interface at R_2 yet.

as we recall the correct procedure is to realize that c_1 and c_2 may be different in each región

region 1 $R_1 < r < R_2$
region 2 $R_2 < r < R_3$

We have found c_2 for region 1. Let us find c_2 for región 2 from the other boundary condition.

$$\phi(R_3) = V_0 = c_1'(\ln R_3 + c_2')$$

$$V_0 = c_1' \ln R_3 + c_1' c_2'.$$

$$\frac{V_0 - c_1' \ln R_3}{c_1'} = c_2'.$$

$$\therefore \phi = c_1' \ln r + (V_0 - c_1' \ln R_3)$$

To find c_1' we first require ϕ to be continuous at $r = R_2$.

$$c_1(\ln \cancel{R_2} - \ln R_1) = c_1' \ln \cancel{R_2} + V_0 - c_1' \ln R_3$$

$$c_1(\ln R_2/R_1) = c_1' \ln (R_2/R_3) + V_0$$

$$c_1 \ln \frac{R_2}{R_1} - c_1' \ln R_2/R_3 = V_0$$

We now have a single equation in two unknowns. Our other equation will come from requiring $\underline{D}$ to be continuous at $r = R_2$. However, we must first find $\underline{E}$

$$\phi = \begin{cases} c_1 \ln(r/R_1) & \text{region 1.} \\ c_1' \ln(r/R_3) + V_0 & \text{region 2.} \end{cases}$$

$$E = -\nabla\phi = -\frac{\partial\phi}{\partial r}$$

$$\underline{E} = \begin{cases} c_1\left(-\frac{1}{r}\right) & \underline{a_r} \quad \text{region I} \\ \\ c_1'\left(-\frac{1}{r}\right) & \underline{a_r} \quad \text{region II} \end{cases}$$

$$\underline{D} = \begin{cases} -c_1\epsilon_1/r \ \underline{a_r} & \text{region I} \\ -c_1'\epsilon_2/r \ \underline{a_r} & \text{region II} \end{cases}$$

requiring $\underline{D}(R_2)$ to be continuous gives.

$$c_1\epsilon_1 = c_1'\epsilon_2$$

$$c_1' = c_1\frac{\epsilon_1}{\epsilon_2}$$

$$c_1 \ln\frac{R_2}{R_1} - c_1' \ln\frac{R_2}{R_3} = V_0 \quad \text{becomes}$$

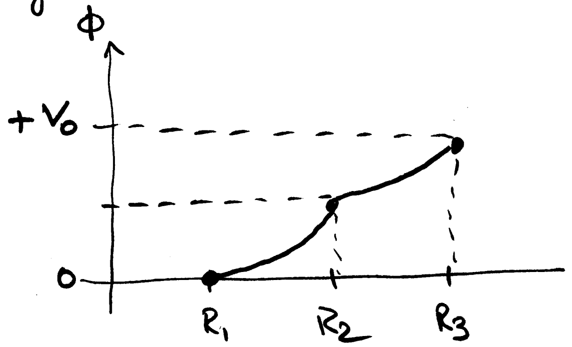
$$c_1 \ln\frac{R_2}{R_1} - c_1\frac{\epsilon_1}{\epsilon_2}\ln\frac{R_2}{R_3} = V_0.$$

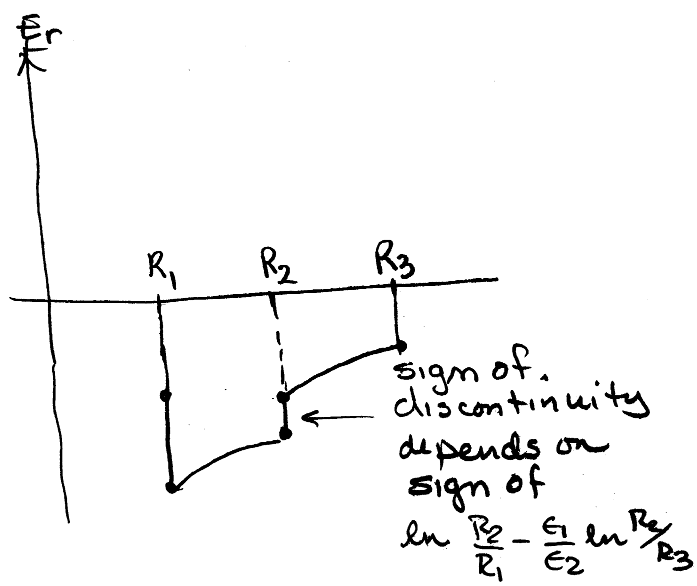
$$c_1 = \frac{V_0}{\ln\frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2}\ln\frac{R_2}{R_3}}$$

$$c_1' = \frac{\frac{\epsilon_1}{\epsilon_2}V_0}{\ln\frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2}\ln\frac{R_2}{R_3}}$$

$$\therefore \quad E = \begin{cases} \dfrac{-\frac{V_0}{r}\,\underline{a_r}}{\ln\frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2}\ln\frac{R_2}{R_3}} & \text{region 1} \\[2ex] \dfrac{-\frac{\epsilon_1}{\epsilon_2}\frac{V_0}{r}\,\underline{a_r}}{\ln\frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2}\ln\frac{R_2}{R_3}} & \text{region 2} \end{cases}$$

Plotting these results





(b) To find capacitance we need the surface charge on each plate. But

$$\underline{n}\cdot(\underline{D}_2 - \underline{D}_1) = \rho_f \quad \text{the free charge density}$$

There is no field outside the capacitor, so let's find the charge density at the inner conductor.

$$\underline{a_r}\cdot\left[\left(\overbrace{\frac{-\epsilon_1\frac{V_0}{R_1}\,\underline{a_r}}{\ln\frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2}\ln\frac{R_2}{R_3}}}^{D_2} - 0\right)\right] = \rho_f.$$

$$\therefore \quad \rho_f = \frac{-\epsilon_1\frac{V_0}{r}}{\left[\ln\frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2}\ln\frac{R_2}{R_3}\right]}$$

this is the charge density. The total charge is then.

$2\pi R_1 l \rho_f$. where l is the length of this capacitor.

or $$Q_1 = \frac{-2\pi\epsilon_1 R_1 \frac{V_0}{\cancel{R_1}}}{\left(\ln\frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2}\ln\frac{R_2}{R_3}\right)}$$

$$= -\frac{2\pi\epsilon_1 V_0}{\left(\ln\frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2}\ln\frac{R_2}{R_3}\right)}$$

The charge on both surfaces is the same so.

$$C = \frac{Q}{V} = \frac{\dfrac{2\pi\epsilon_1 V_0}{\left(\ln\frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2}\ln\frac{R_2}{R_3}\right)}}{V_0}$$

$$\therefore C = \frac{2\pi\epsilon_1 l}{\ln\frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2}\ln\frac{R_2}{R_3}}$$

in summary

$$\phi = \frac{V_0 \ln\left(\frac{r}{R_1}\right)}{\ln \frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2} \ln \frac{R_2}{R_3}} \quad \text{region 1.}$$

$$\phi = \frac{\frac{\epsilon_1}{\epsilon_2} V_0 \ln\left(\frac{r}{R_3}\right)}{\ln \frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2} \ln \frac{R_2}{R_3}} + V_0$$

$$C = \frac{2\pi\epsilon_1 l}{\ln \frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2} \ln \frac{R_2}{R_3}}$$

$$E = \begin{cases} \dfrac{-\frac{V_0}{r}}{\ln \frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2} \ln \frac{R_2}{R_3}} & \text{region 1.} \\[2ex] \dfrac{-\frac{\epsilon_1}{\epsilon_2}\frac{V_0}{r}}{\ln \frac{R_2}{R_1} - \frac{\epsilon_1}{\epsilon_2} \ln \frac{R_2}{R_3}} & \text{region 2} \end{cases}$$